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STABILITY AND MULTI-MODELS IN MODEL REFERENCE ADAPTIVE SYSTEMS.(U)

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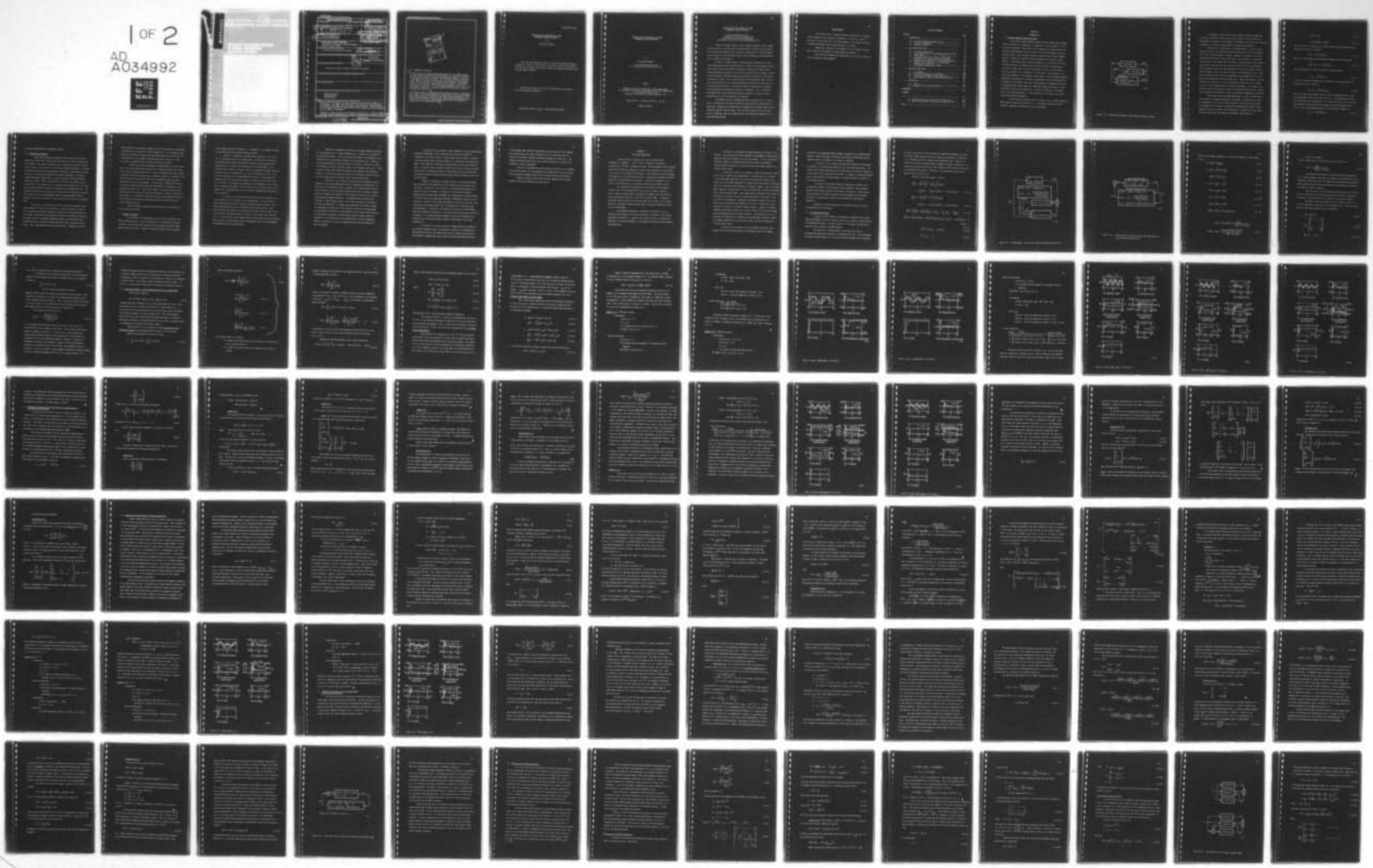

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A method is found, based on controllability conditions of a particular MRAS subsystem state representation, to produce a reduced-order reference model

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20. ABSTRACT (continued)

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A new MRAS structure containing multiple reference models is introduced. We develop a relationship between the structure and a particular dynamic game, comparing the equilibrium of the MRAS and the dynamic game solution obtained assuming a Nash strategy. The stability properties of the 2-model case are examined, under the assumption that each adaptation subsystem is of the hyperstable design.

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STABILITY AND MULTI-MODELS IN MODEL REFERENCE ADAPTIVE SYSTEMS

by

Dale Keith Barbour

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STABILITY AND MULTI-MODELS IN MODEL
REFERENCE ADAPTIVE SYSTEMS

BY

DALE KEITH BARBOUR

B.S., Brown University, 1967
M.S., Carnegie-Mellon University, 1969

THESIS

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Thesis Advisor: Professor William R. Perkins

Urbana, Illinois

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Dale Keith Barbour, Ph.D.
Coordinated Science Laboratory and
Department of Electrical Engineering
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CHAPTER 1

INTRODUCTION

1.1 The Model-Reference Adaptive System

The field of model-reference adaptive systems (MRAS) has developed during the 1960's as an approach to the design of control systems. It provides an alternative to the optimal control approach, a viewpoint which has received the predominant attention from control system researchers during the past decade. The term model reference adaptive control system (MRACS) refers to a variety of system structures which have the following common features. The desired control system performance is specified by the state or output response of a MRACS subsystem which we denote the M-subsystem. The control system itself is part of the MRACS structure, and we denote it as the S-subsystem. It is designed to accept either on-line changes to its controller parameters or an augmented input signal. The S-subsystem parameter adaptation or augmented input signal is generated by a third subsystem which we denote the A-subsystem. This subsystem generates the parameter adjustment or augmented input signal based on a comparison between the output or state responses of the M- and S-subsystems. The overall objective of the MRACS is for the S-subsystem output or state trajectory to converge to the corresponding M-subsystem trajectory.

In this thesis, attention will be focused on the so-called "parallel" MRAS structure, depicted in Figure 1.1-1. The name has resulted from the parallel connection between the M- and S-subsystems.

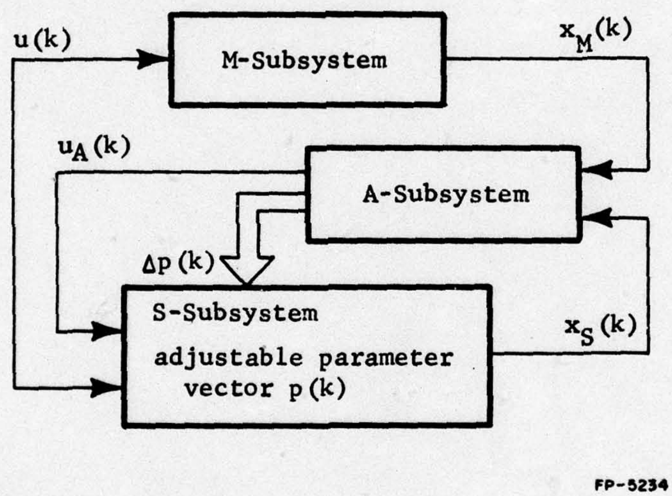

Figure 1.1-1 Discrete-Time Parallel Model Reference Adaptive System

In addition to the control system context, the field of MRAS may be directed toward the system identification problem. In this case, the M-subsystem corresponds to the (possibly controlled) plant while the S-subsystem corresponds to an identification model with adjustable parameters.

Motivation for application of the MRAS concept to control system design is based on the realization that the plant model used in the design of a control system (either by optimal techniques or otherwise) is only an approximate description of the actual plant. For instance, a linear model may be used to approximate what is actually a nonlinear plant; model dynamic order may be lower than that of the actual plant; or the values assigned to model parameters may be only approximations to those in the plant, and the actual parameter values may vary with time in a nondeterministic way. These plant uncertainties may result in unsatisfactory performance of the control system when implemented with the actual plant. Thus a need exists to be able to adapt the controller on-line and automatically when the control system is sensed to be deviating from its intended performance. In the MRAS, the reference model is introduced to specify the desired performance and an adaptation subsystem is provided to direct the necessary compensating adjustments.

The optimal control and MRAS approaches to control system design may be applied as complementary design techniques. The MRAS may be viewed as compensating for the uncertainties ignored in the optimization process. This viewpoint can be simply illustrated by considering the simple linear-quadratic regulator optimal control problem [3]. Consider the linear time-invariant plant model and quadratic performance index defined by:

$$\dot{x} = A_o x + B_o u \tag{1.1-1}$$

$$J = \frac{1}{2} \int_o^\infty (x^T Q x + u^T R u) dt \tag{1.1-2}$$

where $Q \geq 0$, $R > 0$, and (A_o, B_o) provides nominal plant parameter values. The optimal control is given by:

$$u^*(t) = -R^{-1} B_o^T P_o x(t) \tag{1.1-3}$$

where P_o is the unique positive-definite symmetric matrix satisfying:

$$-A_o^T P - PA_o - Q + PB_o R^{-1} B_o^T P = o \tag{1.1-4}$$

Thus the theoretically optimal closed-loop system becomes

$$\dot{x} = (A_o - B_o R^{-1} B_o^T P_o) x \tag{1.1-5}$$

In order to construct a MRACS which complements this optimal design, the M-subsystem would be chosen as:

$$\dot{x}_M = (A_o - B_o R^{-1} B_o^T P_o) x_M \tag{1.1-6}$$

while the actual plant feedback control would be implemented with a time-varying adaptable matrix $P(t)$, where $P(o) = P_o$. The adjustment $\dot{P}(t)$ from the A-subsystem would compensate for the likely possibility that the actual plant matrices (A,B) were not exactly equal to the plant model matrices (A_o,B_o). Thus the S-subsystem (actual controlled plant) appears as

$$\dot{x}_S = (A - BR^{-1} B^T P(t) x_S \tag{1.1-7}$$

with $\dot{P}(t)$ determined by the adaptation design.

1.2 Historical Background

The design philosophy for MRAS has evolved through two distinct phases since about 1960. The earliest published work, which introduced the first phase, is attributed to Whitaker [62], describing an adaptive flight control system. The objective in this phase of MRAS development is to design an adaptation algorithm which minimizes a performance index measuring either the state- or output-error between the M- and S-subsystems. A gradient-related optimization method [44] is usually applied to generate parameter adjustment signals. Other representative work during this phase is found in [25,49,52]. One drawback for the performance index approach is the compromise necessary between state-error convergence rate and MRAS stability. The gradient algorithm gain (or step size for discrete-time problems) is required to be smaller than a system-dependent bound to guarantee MRAS stability. However, this bound also places an upper limit on the state-error convergence rate.

An effort to circumvent the potential instability inherent in this design approach, as well as the necessity of stability analysis, resulted in the second phase, where the design philosophy was to guarantee stability of the MRAS. Among the first published works during this phase were [54,48]. The stable design is obtained by applying Lyapunov's second method [37] to the MRAS structure, resulting in MRAS state-error which is asymptotically stable. Other representative work includes [63,53]. A departure from the

design approach using Lyapunov's method, but still resulting in a guaranteed stable MRAS, was introduced in 1969 by Landau [31]. In this approach, the design is obtained by applying the Popov hyperstability criterion [51] to a reconfigured form of the MRAS. Representative work in this area, done exclusively by Landau and his coworkers, includes [33,35,36].

Several other recent developments in MRAS theory are worth noting here. The stable MRAS design discussed above assumes availability of both the M- and S-subsystem state vectors. Various approaches have been proposed which deal with the case when only the subsystem outputs are available. The state variable filter approach [40], and the augmented signal synthesis approach [46] are two such methods. In the context of MRAS identification, the adaptive observer [6,30,47] has been developed as an extension of the Luenberger observer [41]. This form of MRAS is designed to provide plant identification in conjunction with state estimation. However, question has been raised as to whether the distinction between a standard MRAS identification system and the adaptive observer really goes beyond the issue of system representation [1].

Several excellent surveys, providing extensive bibliographies in this field are [12,39,34].

1.3 Summary of Results

In this thesis, we investigate a series of topics related to the parallel MRAS structure. Except for Section 2.6, the analysis throughout is based on the discrete-time single-input single-output hyperstable MRAS design introduced in [35]. Chapter 2 considers problems for the standard

form of MRAS introduced in Section 1.1. In Chapter 3, a new MRAS structure is introduced which has multiple reference models.

Section 2.1 summarizes the MRAS design presented in [35], which will be referred to here as the Landau MRAS. In Section 2.2 we propose a non-minimal state variable representation, named the Input-Output Delay representation, for the M- and S-subsystems of the Landau MRAS. This representation provides a systematic basis for implementing a MRAS simulation. Although implicit in Landau's work, it is not developed explicitly. The M- and S-subsystems of the Landau MRAS also provide for an assumption of knowledge about the number of plant poles or zeroes located at the origin of the z-plane. By extending the dimension of the parameter vector, we are able to generalize the MRAS to the case where only the dynamic order of the linear plant is assumed known, with no specific knowledge about plant pole or zero locations.

In Section 2.3, we obtain necessary and sufficient conditions for controllability of the I-O Delay state representation. We then show how these conditions may be used to produce a reduced-order M-subsystem. The Landau MRAS assumes both M- and S-subsystems have the same order. This result provides the capability for applying the Landau design to MRAS control problems where the reference model is chosen to be of lower order than the plant. In such a case, we have demonstrated by simulation that the S-subsystem adapts itself to a system whose order is the same as that of the M-subsystem by producing pole-zero cancellations.

Section 2.4 considers the problem of providing asymptotic stability for the parameter-error in MRAS identification, a property not inherent in the hyperstable design. First we develop some geometric and algebraic representations for the equilibrium subspace of the parameter-error. Based on these representations we find necessary and sufficient conditions for which this subspace consists of only the origin of the parameter-error space. Since it turns out that these conditions depend on the unknown plant parameters, however, we conclude that previously published sufficient condition results for this problem are the practical limit in actual MRAS design.

In Section 2.5, we examine a practical problem relating to hyperstability of the Landau MRAS when applied to parameter identification. In this case, the M-subsystem (plant) parameters are assumed unknown, and so a direct solution to the hyperstable design problem does not appear possible. Landau has suggested an approach to this difficulty which does not resolve the potential for MRAS instability. We suggest another approach which introduces a time-varying property to the linear subsystem of the MRAS in its Popov configuration. We suggest some approaches which might be taken to investigate the stability of this modified version of the Landau MRAS. In concluding, we show that by substituting a series-parallel MRAS structure for the parallel structure, the hyperstable design no longer depends on parameters in the M-subsystem, thus eliminating this problem altogether. However, one must pay a price, as Landau has shown, when operating in a noisy environment.

In Section 2.6, we consider a class of MRAS which uses discontinuous adaptation algorithms. Work in this area has been done primarily by researchers in the Soviet Union. Using results from the theory of variable structure systems (VSS), we prove stability of a particular continuous time MRAS. This viewpoint is different from that taken in previously published work. In previous cases, Lyapunov's method has been used to prove stability. Here we show an equivalence between the MRAS problem and a VSS measurable disturbance rejection problem, and then apply existing stability results for the VSS problem.

Chapter 3 introduces a new MRAS structure, the multi-model MRAS. This particular form of MRAS has not been treated in the literature up to this time. In Section 3.1 the structure is introduced and motivation for interest in such a structure is presented. We also jointly develop a regulator-type two-model MRAS (2M-MRAS) and a dynamic game, the goal being to provide a basis for examining a possible correspondence between the two structures. As part of this joint development, we introduce alternate viewpoints which the players might adopt in obtaining their reference models (M_i-subsystems). In Section 3.2 we examine the stability of the 2M-MRAS where each adaptation subsystem is of the hyperstable Landau-type considered in Chapter 2. By transforming the 2M-MRAS to the Popov configuration we identify how its structure differs from the standard MRAS in that same configuration.

In Section 3.3 we jointly develop a dynamic game and a 2M-MRAS of the quadratic regulator type. We establish a procedure for comparing the equilibrium solution of the dynamic game and the parameter equilibrium of the 2M-MRAS by comparing the actual closed-loop Nash equilibrium solution

of the dynamic game with the equilibrium structure obtained in the 2M-MRAS. For the particular case under consideration, we conclude that the two equilibrium solutions represent different strategies and objectives. The 2M-MRAS equilibrium structure is then obtained explicitly in terms of the parameters of the M_i-subsystems.

Section 3.4 considers the quasi-symmetric character of the 2M-MRAS, and we obtain results for the regulator-type 2M-MRAS which define how deviations from perfect symmetry effect the S-subsystem parameter equilibrium.

Chapter 4 contains conclusions and some suggestions for further research on the topics addressed in this thesis.

CHAPTER 2

THE SINGLE MODEL MRAS

In this chapter we consider the class of parallel MRAS introduced in Chapter 1. This class is defined by a discrete-time SISO-LTV $n^{\underline{th}}$-order subsystem S, whose (2N+1) parameters vary according
time SISO-LTV $n^{\underline{th}}$-order subsystem S, whose (2N+1) parameters vary according
to the output of an adaptation subsystem A. The adaptation subsystem measures and processes M- and S-subsystem variables to determine S-subsystem parameter changes at equally-spaced time intervals. The goal of subsystem A is to drive to zero state-error, and possibly parameter-error between M and S.

The terminology "single model" is introduced in this chapter to contrast this class of systems with a class to be studied in Chapter 3. Here, a single reference model is assumed in the MRAS, which specifies the single performance objective or goal for the overall MRAS. In Chapter 3, we consider the case where two reference models are assumed, each of which specifies a different, possibly conflicting, performance objective or goal for the overall MRAS.

The adaptation subsystem A considered in Sections 2.1-2.5 is continuous with respect to all input variables, in contrast to the problem considered in Section 2.6. There we will study the class of systems where subsystem A is defined according to a difference equation with discontinuous right-hand side.

In Section 2.1 we present the MRAS mathematical structure to be studied. This structure has been proposed by Landau [35] for application to MRAS identification problems in a noisy environment. Its most significant property, viewed as a nonlinear discrete-time system, is guaranteed global asymptotic stability in state-error space, and global stability in parameter-error space.

In Sections 2.2 and 2.3, we propose a state representation, which we call the I-O Delay state representation, and show that it is inherent in the Landau MRAS algorithm. This state representation enables one to express the Landau MRAS algorithm in a compact form, and provide a systematic basis for implementing the algorithm on a digital computer. Next, the Landau MRAS algorithm is generalized to a less restricted form, called the Extended Landau MRAS Algorithm (ELMA). This generalization eliminates the assumption by Landau of the number of poles or zeroes occurring at the origin of the z-plane in the M-subsystem. Necessary and sufficient conditions are then found for controllability and observability of the I-O delay state representation, in terms of the system transfer function. Lastly, we obtain an algebraic relationship between the I-O delay state and an arbitrary minimal state representing the same transfer function. The motivation here is to determine the feasibility of expressing the Landau MRAS algorithm in terms of a minimal dimension (N) state vector, rather than the 2N-dimensional I-O Delay State vector.

In Section 2.4 we consider the relationship between the input sequence $u(k)$ and the equilibrium for the parameter-error of the MRAS.

This topic is of importance when the MRAS is applied to the identification problem. Results relating to necessary and sufficient conditions for the parameter-error to converge to the origin are obtained.

In Section 2.5 we consider some practical difficulties relating to the stability of the MRAS when applied to the identification problem. The positive-real design requirement for the MRAS cannot be directly realized due to the nature of the identification problem. We suggest a design modification to the Landau MRAS and describe methods to analyze stability properties of this modified version.

In Section 2.6 the MRAS adaptation algorithm is assumed to be a discontinuous function of the measured system variables. Both parameter adaptation and signal synthesis formulations are presented. Using results on the stability of variable structure systems designed for disturbance rejection, we establish the stability of the MRAS with discontinuous adaptation.

Appendix A contains definitions and notations which are used in the development of this chapter.

2.1 The Landau MRAS Design

In this section we present a mathematical statement of the Landau MRAS design [35], in order to develop a basis in the reader for understanding the fundamental problem. From this basis we will be ready to examine more specific issues related to the MRAS problem.

The basis for his design method is the result by Popov on necessary and sufficient conditions for hyperstability [50]. By transforming the parallel MRAS structure to the linear-nonlinear feedback form considered

by Popov, he was able to find a particular adaptation subsystem A so that the overall MRAS satisfies the Popov asymptotic hyperstability conditions; i.e., the MRAS state-error asymptotically approaches zero. Figure 2.1-1 depicts the particular parallel MRAS considered by Landau, and Figure 2.1-2 depicts that system transformed to the linear-nonlinear structural form used for hyperstability analysis. Definition of the equations in each block appears later in this section.

Define the following (n+m+1) vectors:

$$\bar{x}_M^T(k) = [x_M^{o^T} \mid \bar{x}_M^{1^T}] = [x_M^{o^T} \mid x_M^{1^T} \mid u(k)]$$

$$= [y_M(k-n) \cdots y_M(k-1) \mid u(k-m) \cdots u(k-1) \mid u(k)] \qquad (2.1\text{-}1)$$

$$\bar{x}_S^T(k) = [x_S^{o^T} \mid \bar{x}_S^{1^T}] = [x_S^{o^T} \mid x_S^{1^T} \mid u(k)]$$

$$= [y_S(k-n) \cdots y_S(k-1) \mid u(k-m) \cdots u(k-1) \mid u(k)] \qquad (2.1\text{-}2)$$

$$\bar{p}_M^T = [a_M^T \mid \bar{b}_M^T] = [a_M^T \mid b_M^T \mid b_{M_o}] = [a_{M_n} \cdots a_{M_1} \mid b_{M_m} \cdots b_{M_1} \mid b_{M_o}] \qquad (2.1\text{-}3)$$

$$\bar{p}^T(k) = [a^T(k) \mid \bar{b}^T(k)] = [a^T(k) \mid b^T(k) \mid b_o(k)] = [a_n(k) \cdots a_1(k) \mid b_m(k) \cdots \mid b_o(k)] \qquad (2.1\text{-}4)$$

$$e^T(k) = [\epsilon(k-n) \cdots \epsilon(k-1)] \qquad (2.1\text{-}5)$$

$$c^T = [c_n \cdots c_1] \qquad (2.1\text{-}6)$$

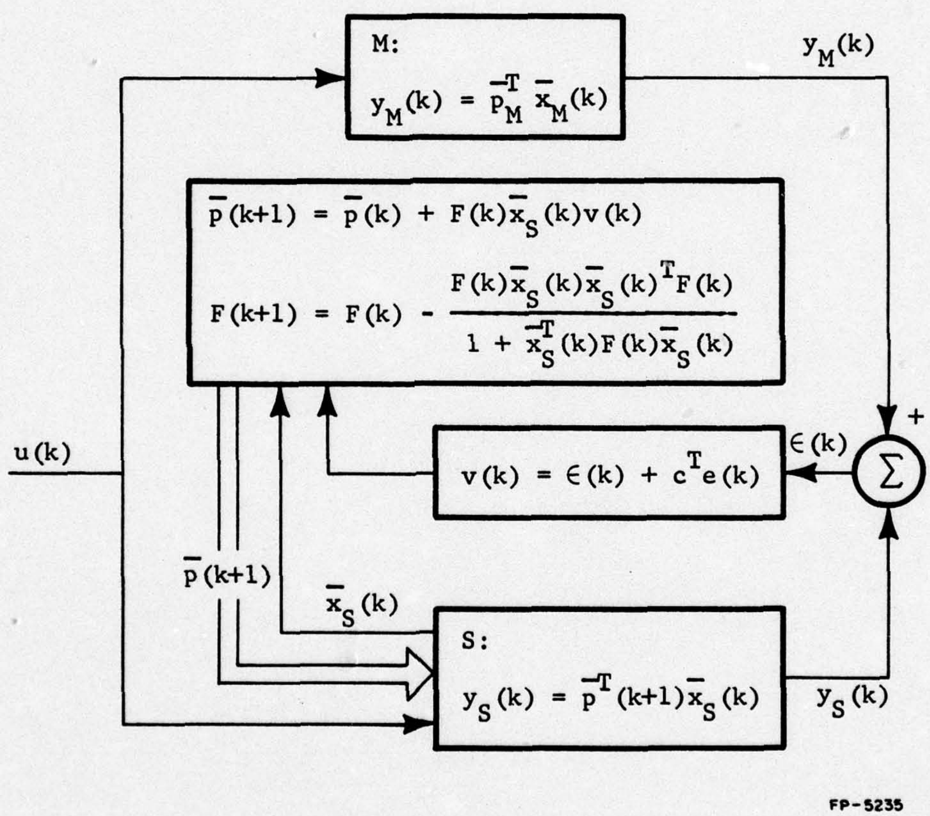

Figure 2.1-1 Landau MRAS-- Discrete-Time Linear SISO Subsystems M and S

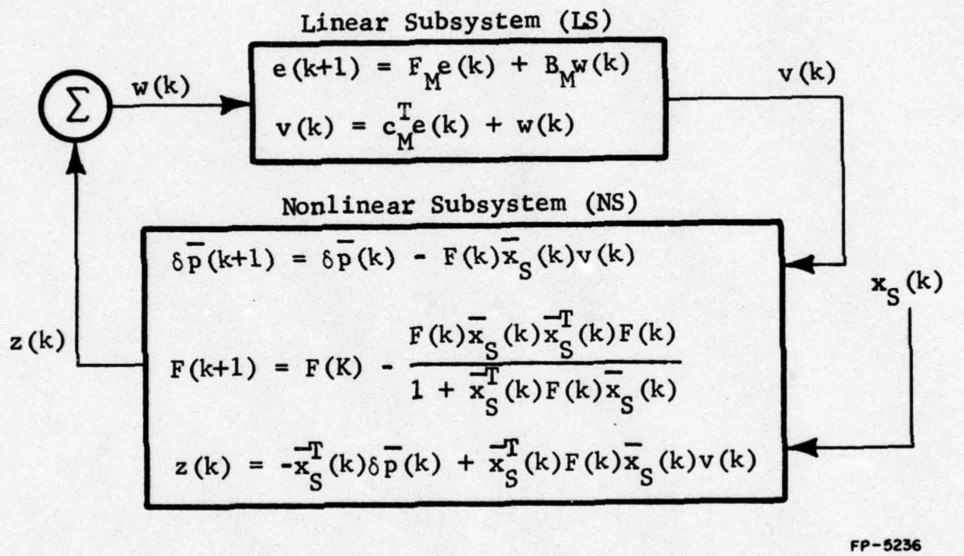

Figure 2.1-2 Transformation of Landau Parallel MRAS Structure for Popov Hyperstability Analysis

Then the three MRAS subsystems M, S, and A are defined to be (See Figure 2.1-1):

$$\text{M:} \quad y_M(k) = \bar{p}_M^T \, \bar{x}_M(k) \tag{2.1-7}$$

$$\text{S:} \quad y_S(k) = \bar{p}^T(k+1) \, \bar{x}_S(k) \tag{2.1-8}$$

$$y_S^o(k) = \bar{p}^T(k) \, \bar{x}_S(k) \tag{2.1-9}$$

$$\text{A:} \quad \epsilon(k) = y_M(k) - y_S(k) \tag{2.1-10}$$

$$\epsilon^o(k) = y_M(k) - y_S^o(k) \tag{2.1-11}$$

$$v(k) = \epsilon(k) + c^T e(k) \tag{2.1-12}$$

$$v^o(k) = \epsilon^o(k) + c^T e(k) \tag{2.1-13}$$

$$\bar{p}(k+1) = \bar{p}(k) + [F(k) \, \bar{x}_S(k)] v(k) \tag{2.1-14}$$

$$= \bar{p}(k) + [F(k) \, \bar{x}_S(k)] \frac{v^o(k)}{1 + \bar{x}_S^T(k) \, F(k) \, \bar{x}_S(k)}$$

$$F(k+1) = F(k) - \frac{[F(k) \, \bar{x}_S(k)][F(k) \, \bar{x}_S(k)]^T}{1 + \bar{x}_S^T(k) \, F(k) \, \bar{x}_S(k)} \tag{2.1-15}$$

$F(0) > 0$; symmetric

It may be shown that (2.1-12) and (2.1-13) are related by:

$$v(k) = \frac{v^{o}(k)}{1 + \bar{x}_S^T(k)\, F(k)\, \bar{x}_S(k)} \tag{2.1-16}$$

The variables which appear both without and with superscript o (e.g., $y_S(k)$, $y_S^o(k)$) are referred to by Landau as a posteriori and a priori variables, respectively, since the superscripted variables are generated before updating $\bar{p}(k)$ while the non-superscripted variables are generated after $\bar{p}(k)$ is updated to $\bar{p}(k+1)$.

Note also that no recursive expressions are written for the $\bar{x}(k)$ vectors, although these expressions should be provided for completeness. This aspect of the algorithm will be dealt with in the next section.

The Popov structure in Figure 2.1-2 is obtained from (2.1-7 - 2.1-15), where $\delta\bar{p}(k) = \bar{p}_M - \bar{p}(k)$ is the parameter-error. The state-realization of the linear subsystem (LS) is defined by:

$$C_M^T = [a_M^T + c^T] \tag{2.1-17}$$

$$F_M = \begin{bmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & & & 0 \\ 0 & & & 1 \\ a_{M_n} & & \cdots & a_{M_1} \end{bmatrix} \tag{2.1-18}$$

$$B_M^T = [0 \cdots 0\ 1]; \tag{2.1-19}$$

i.e., this is a phase-canonic (completely controllable) realization.

In developing the necessary and sufficient conditions for hyperstability, Popov assumes the nonlinear subsystem satisfies an inequality constraint:

$$\sum_{k=o}^{K} z(k)\ v(k) \geq -\gamma_o^2 \tag{2.1-20}$$

for all K, γ_o^2 a positive real constant.

However, (NS) is actually a linear time-varying block with respect to the variables v(k) and z(k), with state vector $\delta\overline{p}(k)$ and time-varying coefficients computed from $\overline{x}_S(k)$ and F(k). Thus, a sufficient condition for (NS) to satisfy (2.1-20) is that (NS) satisfies the conditions of the positive real lemma extended to time-varying discrete systems [36].

The transfer function of (LS) is:

$$LS(z) = 1 + \frac{\sum_{i=1}^{n} (a_{M_i} + c_i) z^{n-i}}{z^n - \sum_{i=1}^{m} a_{M_i} z^{n-i}} \tag{2.1-21}$$

It is necessary and sufficient that $c \in R^n$ be chosen so that LS(z) is a discrete positive real transfer function [16]. This discrete positive real concept is defined in Section 2.5. This is not a trivial problem, and unless proper attention is paid to this design phase, the algorithm will necessarily lose its hyperstability property. Setting $c = -a_M$ clearly satisfies the positive real condition. However, for the identification problem, a_M corresponds to part of the unknown plant parameter vector, so setting $c = -a_M$ is not possible. Landau suggests obtaining an initial estimate $\hat{a}$ for the S-subsystem parameter a(k) in (2.1-4), and then choose $c = -\hat{a}$.

Although this approach does not guarantee satisfaction of the positive real condition, Landau claims it has not failed him in achieving state-error convergence to zero. In Section 2.3 we add another condition, not addressed by Landau, which c must satisfy, based on assumptions made by Popov. In Section 2.5 we consider in more detail the effects of choosing the vector c.

2.2 Input-Output Delay - A State Representation for the Landau MRAS

The difference equation:

$$y(k) = \overline{p}^T \overline{x}(k) = \sum_{i=1}^{n} a_i \, y(k-i) + \sum_{i=o}^{m} b_i \, u(k-i) \tag{2.2-1}$$

representing either subsystem M or S of the Landau MRAS in Section 2.1 may be expressed both as a transfer function and in state variable form. In this section we find the transfer function for arbitrary (n,m), and express the difference equation in a state variable form. This form provides a basis from which we will later analyze the MRAS subsystems, it completes the dynamic expression of the MRAS algorithm, and provides a systematic basis for implementing the algorithm on digital computer.

The structure of (2.2-1) suggests the name input-output delay difference equation, and henceforth we will refer to it by that name.

Taking the Z-transform of (2.2-1) we obtain:

$$(1 - \sum_{i=1}^{n} a_i z^{-i}) \, Y(z) = (\sum_{i=o}^{m} b_i z^{-i}) \, U(z) \tag{2.2-2}$$

Then the transfer function is:

$$
H(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{i=o}^{m} b_i z^{-i}}{1 - \sum_{i=1}^{n} a_i z^{-i}}
= \begin{cases}
\dfrac{z^{(n-m)} \sum_{i=o}^{m} b_i z^{m-i}}{z^n - \sum_{i=1}^{n} a_i z^{n-i}} & \text{if } n > m \\[2ex]
\dfrac{\sum_{i=o}^{n} b_i z^{n-i}}{z^n - \sum_{i=1}^{n} a_i z^{n-i}} & \text{if } n = m \\[2ex]
\dfrac{\sum_{i=o}^{m} b_i z^{m-i}}{z^{(m-n)} [z^n - \sum_{i=1}^{n} a_i z^{n-i}]} & \text{if } n < m
\end{cases}
\qquad (2.2\text{-}3)
$$

We note that, when $n \neq m$, either:

1) at least $(n-m)$ zeroes exist at the origin out of a maximum of $(n-1)$ possible zeroes, or
2) at least $(m-n)$ poles exist at the origin out of a total of m poles

Denote $N = \max(n,m)$ as the order of the transfer function. We may now write a single expression for $H(z)$:

$$H(z) = \frac{\sum_{i=o}^{m} b_i \, z^{N-i}}{z^N - \sum_{i=1}^{n} a_i \, z^{N-i}} \tag{2.2-4}$$

Either n or m may be less than N. However, there is no loss in generality by extending $a^T = [a_n \cdots a_1]$ or $b^T = [b_m \cdots b_1]$ to dimension N, and imposing either $[a_N \cdots a_{n+1}] = 0$ or $[b_N, \cdots, b_{m+1}] = 0$, whichever is appropriate. Then the system:

$$y(k) = \sum_{i=1}^{N} (a_i y(k-i) + b_i u(k-i)) + b_o u(k) \tag{2.2-5}$$

$$H(z) = \frac{\sum_{i=o}^{N} b_i z^{N-i}}{z^N - \sum_{i=1}^{N} a_i z^{N-i}} = \frac{\sum_{i=1}^{N} (b_i + a_i b_o) z^{N-i}}{z^N - \sum_{i=1}^{N} a_i z^{N-i}} + b_o \tag{2.2-6}$$

is equivalent to (2.2-1, 2.2-4), with appropriate coefficients set to zero. We will refer to (2.2-5, 2.2-6) as the extended version of the system (2.2-1, 2.2-4).

Consider now the 2N-dimensional state variable definition:

$$x^T(k) = [x^{o^T}(k) \mid x^{1^T}(k)] = [y(k-N) \cdots y(k-1) \mid u(k-N) \cdots u(k-1)] \tag{2.2-7}$$

Then a state variable realization of the extended system (2.2-5, 2.2-6) is:

$$x(k+1) = \tilde{F}\, x(k) + \tilde{G}\, u(k) \tag{2.2-8}$$

$$y(k) = p^T x(k) + b_o\, u(k) \tag{2.2-9}$$

where

$$\tilde{F} = \left[\begin{array}{cc} F & \hat{F}(1) \\ \hline 0 & I_N^1 \end{array}\right] \tag{2.2-10}$$

$$\tilde{G}^T = [G_1^T \mid G_2^T] = [0 \cdots 0 b_o \mid 0 \cdots 01] \tag{2.2-11}$$

$$p^T = [a^T \mid b^T] = [a_N \cdots a_1 \mid b_N \cdots b_1] \tag{2.2-12}$$

The matrices F, $\hat{F}(1)$, and I_N^1 are fully defined in Appendix A.

The validity of (2.2-8 - 2.2-12) as a 2N-dimensional realization of (2.2-5, 2.2-6) may be established by inspection of the state definition, and recognizing that $x_N^o(k+1) = y(k)$. Clearly y(k) is identically defined in (2.2-5) and (2.2-9). We will call this state representation the <u>I-O Delay State Representation</u>.

The motivation for extending both a and b to R^N from R^n and R^m, respectively, lies in the intention to remove from the Landau algorithm a potentially restrictive assumption which he implicitly makes. The system (3.3-1, 2.2-3) assumes, not only the denominator and numerator dimensions (i.e., basic system structure), but further that both subsystems M and S have either (n-m) zeroes at the origin (when n > m), or (m-n) poles at the

origin (when $m > n$). By extending the parameter vectors a and b to R^N, we choose to assume basic structure only, allowing the identification process to determine whether or not there are poles or zeroes at the origin. This is accomplished by observing whether either $[a_N(k) \cdots a_{n+1}(k)]$ or $[b_N(k) \cdots b_{m+1}(k)]$ converges to zero as a result of the adaptation process. We will refer to the algorithm using the extended parameter set as the Extended Landau MRAS Algorithm (ELMA).

The I-O Delay State Representation and the associated realization for (2.2-6) provide a systematic method for recursively generating x(k). We make use of this representation in stating the Landau MRAS algorithm in its extended form (ELMA):

$$\text{M:} \quad x_M(k+1) = \tilde{F}_M \, x_M(k) + \tilde{G}_M \, u(k) \tag{2.2-13}$$

$$y_M(k) = p_M^T \, x_M(k) + b_{M_o} \, u(k) \tag{2.2-14}$$

$$\text{S:} \quad x_S(k+1) = \tilde{F}(k+1) \, x_S(k) + \tilde{G}(k+1) \, u(k) \tag{2.2-15}$$

$$y_S(k) = p^T(k+1) \, x_S(k) + b_o(k+1) \, u(k) \tag{2.2-16}$$

$$y_S^o(k) = p^T(k) \, x_S(k) + b_o(k) \, u(k) \tag{2.2-17}$$

A: This subsystem remains unchanged, except that we define state-error:

$$e(k+1) = I_N^1 \, e(k) + G_2 \, \epsilon(k), \tag{2.2-18}$$

$\bar{x}_S(k)$ is formed by appending u(k), the current input, and $\bar{p}(k)$ corresponds now to the extended parameter set. An alternate dynamic equation for e(k), obtained directly from (2.2-13, 2.2-15) is:

$$e(k+1) = F_M\, e(k) + G_2\, \bar{x}_S^T(k)\, \delta\bar{p}(k+1) \qquad (2.2\text{-}19)$$

We conclude this section by presenting simulation results of two examples of the Extended Landau MRAS Algorithm (ELMA). The first example is of a $1^{\underline{st}}$ order problem (3 parameters), the second of a $2^{\underline{nd}}$ order problem (5 parameters). In each example we consider both pulse input and sinusoidal input. Time discretization is taken as 0.1 second. Initial state conditions for the M- and S-subsystems are assumed zero.

<u>Example 2.2-1</u> - $1^{\underline{st}}$ Order Problem

M-subsystem:

a) pole at z = 0.9

b) $b_o = 0.1$

c) b_1 chosen to yield unity gain on $|z| = 1$

Thus $\bar{p}_m^T = [0.9 \,\vdots\, 0,\ 0.1]$

Initial S-subsystem:

a) pole at z = 0

b) S-transfer function numerator = M-transfer function numerator

Thus $\bar{p}^T(0) = [0 \,\vdots\, 0.09,\ 0.1]$

A-subsystem:

a) $F(0) = \text{diag}\ [1000, 1000, 1000]$

b) $c(t) = -a(t)$

Input u(k):

case (a) - pulse with (amplitude, period) = (1,5)

case (b) - sine with (amplitude, period) = (1,4)

Final S-subsystem:	$\bar{p}(10)$
case (a)	$[0.9 \mid 6 \times 10^{-5}, 0.1]$
case (b)	$[0.904 \mid -0.012, 0.11]$

Simulation results are shown in Figure 2.2-1. In both input cases, state-error e(k) converges to zero, in accordance with the hyperstable property of the MRAS. Furthermore, parameter-error, $\delta\bar{p}(k) = \bar{p}_M - \bar{p}(k)$, converges to zero. ■

<u>Example 2.2-2</u> 2^{nd} Order Problem

M-subsystem:

a) poles at z = (0.9, 0.4)

b) zero at z = 0.8

c) $b_o = 0$

d) b_1 chosen to yield unity gain on $|z| = 1$

Thus $\bar{p}_M^T = [-0.36, 1.3 \mid -.24, .3, 0]$

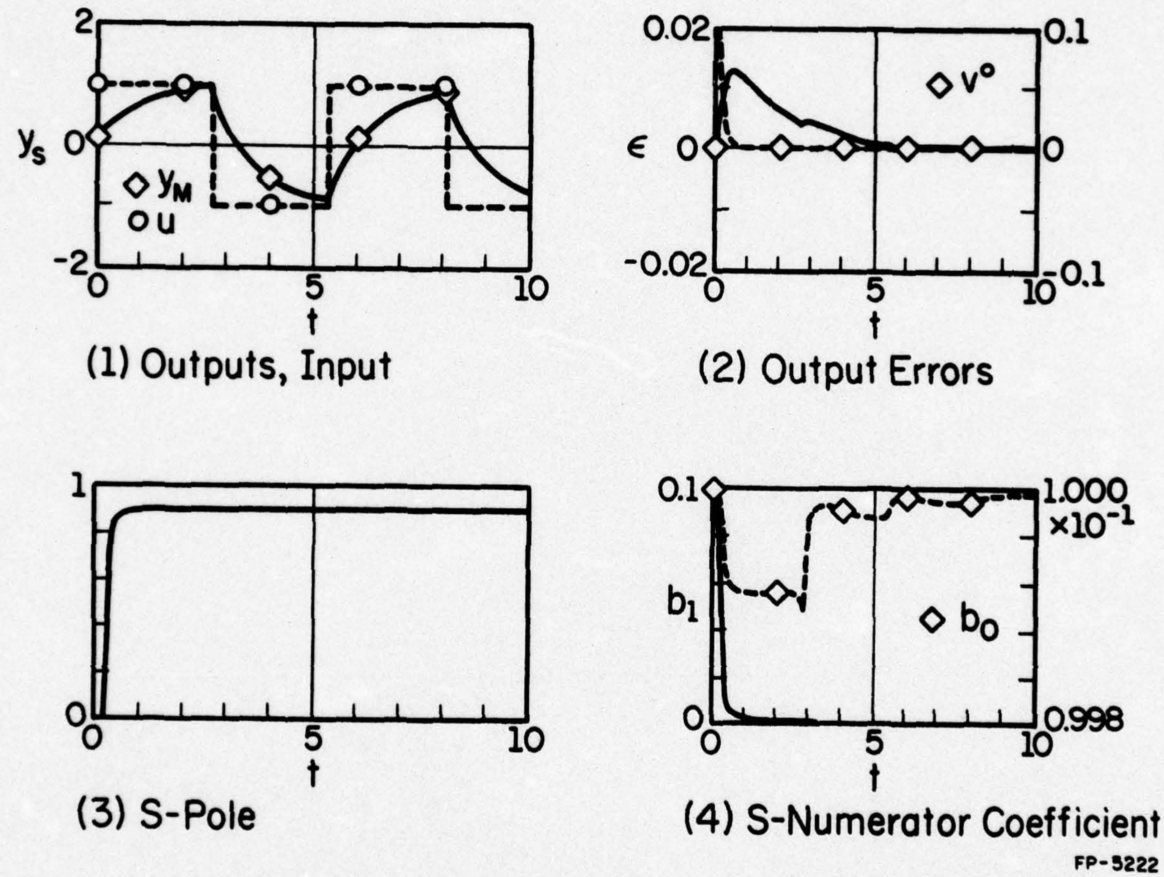

Figure 2.2-1(a) ELMA Example 2.2-1--Case a

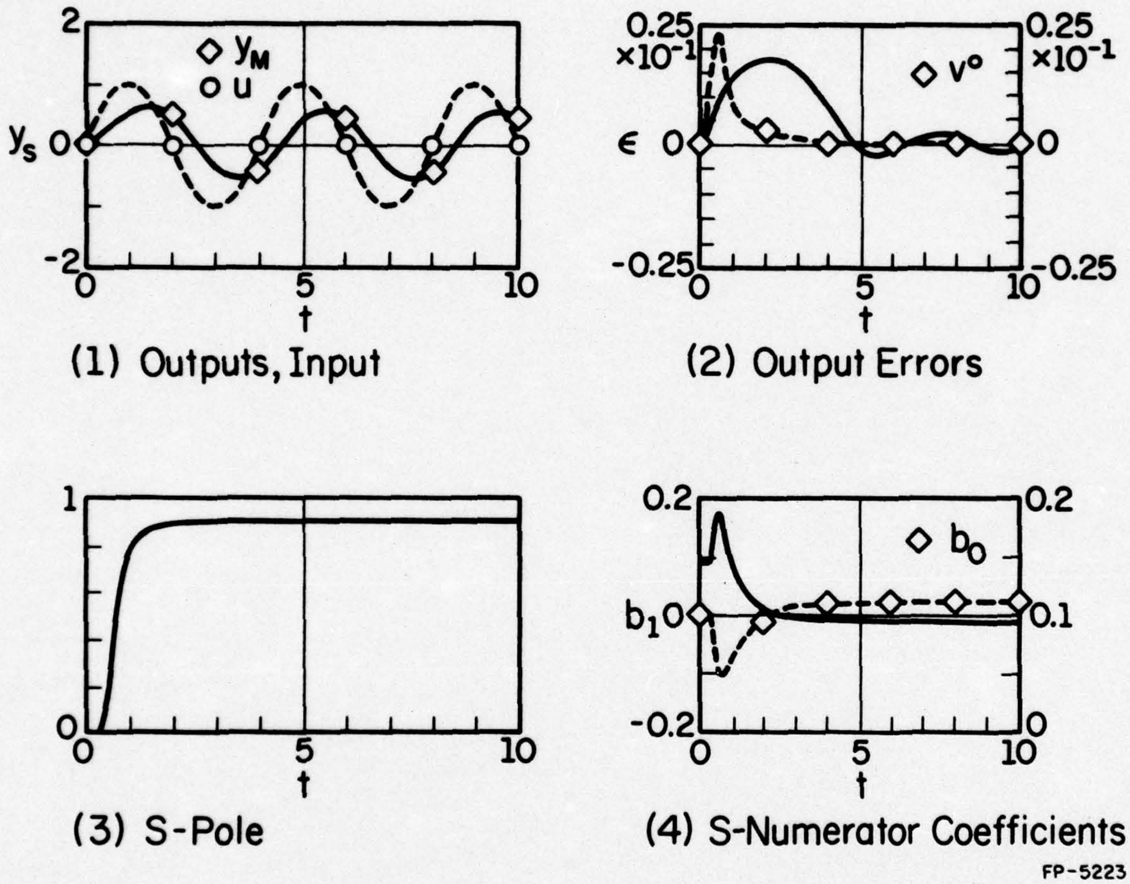

Figure 2.2-1(b) ELMA Example 2.2-1--Case b

Initial S-subsystem:

a) poles at z = (0,0)

b) S-transfer function numerator = M-transfer function numerator

Thus $\bar{p}^T(0) = [0,0 \vdots -0.24, 0.3, 0]$

A-subsystem:

a) F(0) = diag [1000, 1000, 1000, 1000, 1000]

b) c(t) = - a(t)

Input u(k):

case (a) - pulse with (amplitude, period) = (1,5)

case (b) - sine with (amplitude, period) = (1,4)

case (c) - sine with (amplitude, period) = (1,20) + (10,3)

Final S-subsystem:

case	$\bar{p}^T(12)$	(poles, zeroes)
a)	$[-0.356, 1.3 \vdots -0.239, 0.3, 10^{-4}]$	$(.898, .391 \vdots .792)$
b)	$[0.105, 0.754 \vdots -0.263, 0.392, 0.008]$	$(-0.12, 0.874 \vdots 0.66)$
c)	$[-0.361, 1.3 \vdots -0.242, 0.301, 1.6\text{x}10^{-4}]$	$(0.901, 0.4 \vdots 0.801)$

Simulation results are shown in Figure 2.2-2. In all three input cases, the state-error converges to zero. Notice, however, that parameter convergence fails for case (b), where the sine input contains only a single

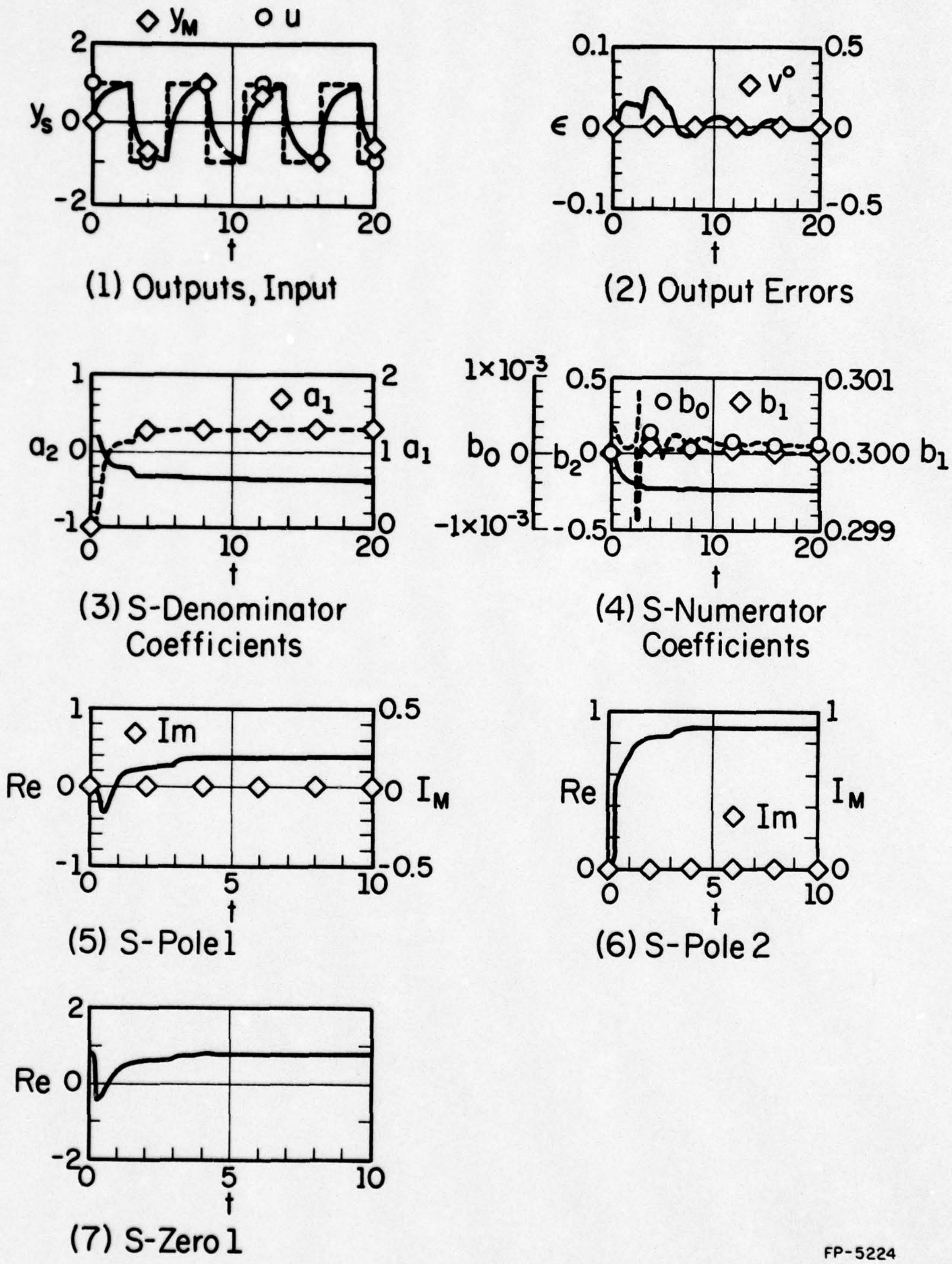

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Figure 2.2-2(a) ELMA Example 2.2-2--Case a

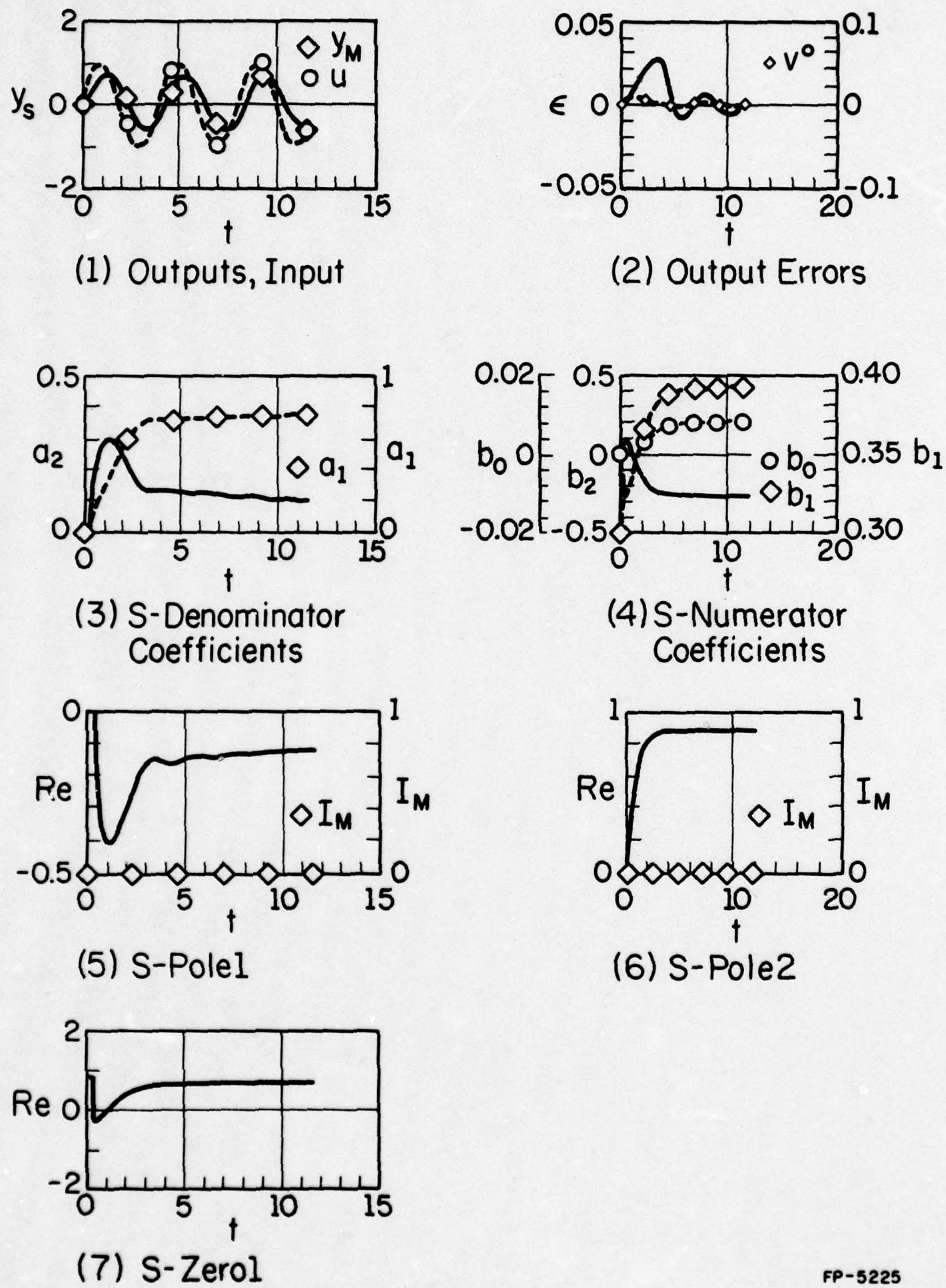

Figure 2.2-2(b) ELMA Example 2.2-2--Case b

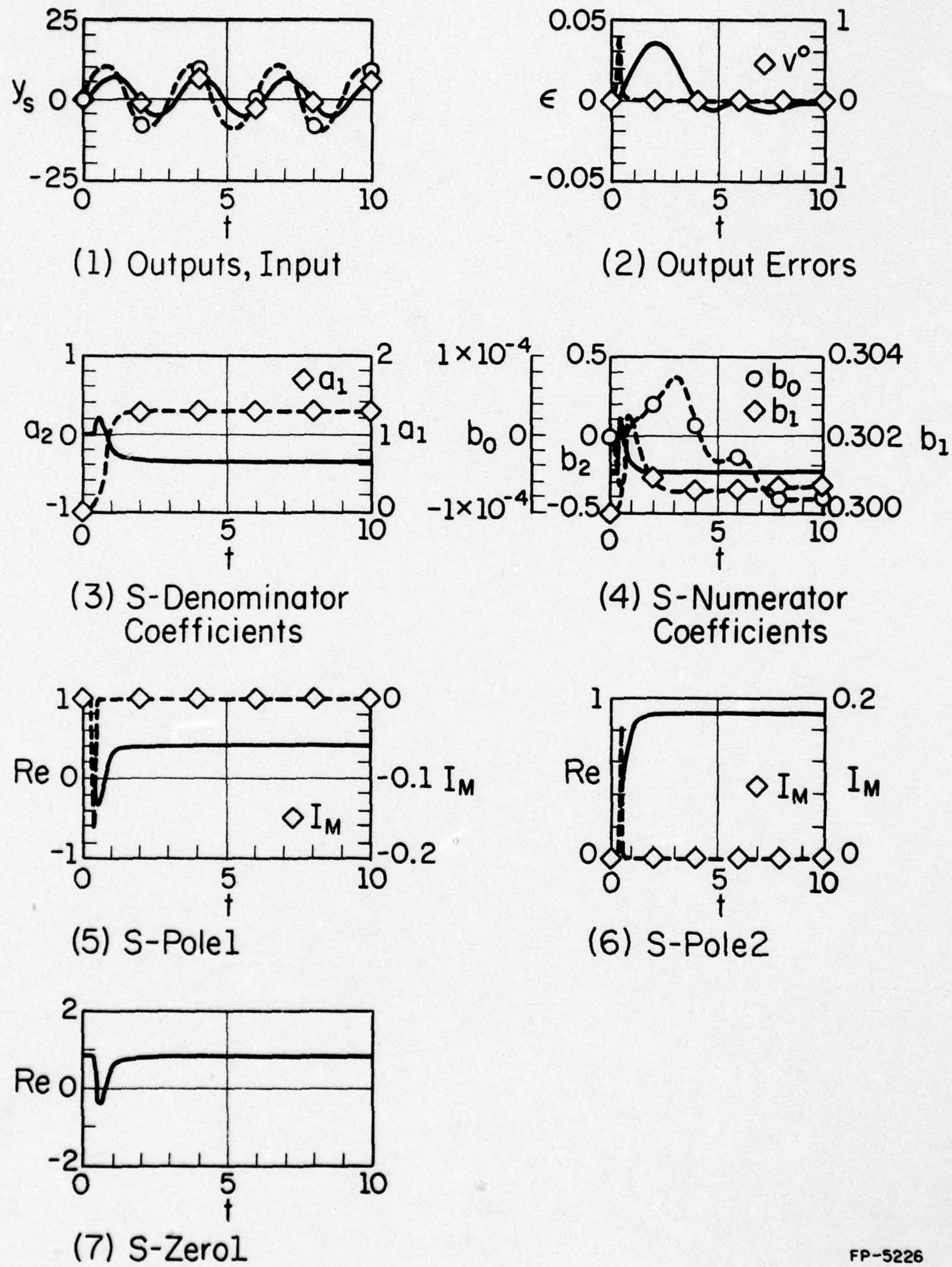

Figure 2.2-2(c) ELMA Example 2.2-2--Case c

frequency. The Landau MRAS design guarantees only that the state-error is asymptotically hyperstable, and claims nothing more than stability (boundedness) for the parameter error vector $\delta\bar{p}(k) = \bar{p}_M - \bar{p}(k)$. In Section 2.4 we investigate this aspect of the MRAS problem in detail. ■

2.3 Structural Properties of the I-O Delay State Representation - Implications for the MRAS

In this section we derive a series of properties relating to the structure of the I-O Delay state representation. First we examine the conditions for controllability and observability of the I-O Delay state representation. This leads to consideration of the effects of pole-zero cancellation (and loss of I-O Delay state controllability) in the M-subsystem on the performance of the Extended Landau MRAS Algorithm (ELMA).

Next we find an algebraic relation between the 2N-dimensional I-O Delay state and an arbitrary N-dimensional state, each of which realizes the same transfer function (2.2-6). This leads to the problem of expressing the I-O Delay state as a function of a minimal state. The motivation here is to determine whether expressing the ELMA in terms of minimal state representations of subsystems M and S will introduce computational economies.

The conditions for controllability and observability of the I-O Delay state realization (2.2-8 - 2.2-12) may be determined by examining the controllability and observability matrices:

$$R_c = [\tilde{G} \quad \tilde{F}\tilde{G} \quad \cdots \quad \tilde{F}^{2N-1}\tilde{G}] \tag{2.3-1}$$

$$R_o = \begin{bmatrix} p^T \\ p^T \tilde{F} \\ \vdots \\ p^T \tilde{F}^{2N-1} \end{bmatrix} \tag{2.3-2}$$

Using (2.2-10 - 2.2-12), R_c and R_o may be expressed:

$$R_c = \left[\begin{array}{cccc|cccc} h(0) & h(1) & \cdots & h(N-1) & h(N) & F\,h(N) & \cdots & F^{N-1}\,h(N) \\ \hline & P_N^o & & & & 0 & & \end{array}\right] \tag{2.3-3}$$

where $h(k) = F^k\, G_1 + \hat{F}(k)\, G_2$, $k = o, \cdots, N$, (2.3-4)

F, $\hat{F}(k)$, P_N^o are defined in Appendix A, G_1 and G_2 in (2.2-11), and

$$R_o = \left[\begin{array}{c|c} F^N & \hat{F}(N) \\ \hline F^{2N} & F^N\,\hat{F}(N) \end{array}\right] \tag{2.3-5}$$

The next 4 lemmas will be required in what follows. The first is a useful and well-known fact from matrix algebra [10].

<u>Lemma 2.3.1</u>

Given a square matrix partitioned into 4 submatrices

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] ,$$

assuming either A_1 or A_4 is nonsingular, then

$$\det(A) = \det(A_1)\cdot\det(A_4 - A_3A_1^{-1}A_2)$$
$$= \det(A_4)\cdot\det(A_1 - A_2A_4^{-1}A_3)$$

■

Lemma 2.3.2

Two arbitrary SISO LTI systems, $S_1 = (C_1,A_1,B_1)$ and $S_2 = (C_2,A_2,B_2)$ are realizations for the same transfer function if and only if

$$C_1A_1^kB_1 = C_2A_2^kB_2 \quad \text{for all } k = 0,1,2,\cdots$$

Proof: The relaxed output response to an input

$$u(k) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}, \quad \{y(0),\ y(1),\ y(2),\ \cdots\},$$

called the input weighting sequence, for each system is:

$$\{0,\ C_1B_1,\ C_1A_1B_1,\ C_1A_1^2B_1,\cdots\} \text{ and } \{0,\ C_2B_2,\ C_2A_2B_2,\ C_2A_0^2B_2,\cdots\},$$

respectively.

The relaxed output response from two equivalent system realizations must be equal. Therefore their respective input weighting sequences must be equal. By definition of the input weighting sequence, two equal input weighting sequences imply two equivalent systems; i.e., they realize the same transfer function. ■

In the next lemma, we consider the SISO N-dimensional system:

$$w(k+1) = F\ w(k) + h(N)\ u(k) \tag{2.3-6}$$

$$y(k) = c^T w(k) + b_o u(k) \tag{2.3-7}$$

where h(N) is defined by (2.3-4), F from Appendix A.2, and c^T arbitrary.

Lemma 2.3.3

The system (2.3-6, 2.3-7) is a completely observable realization of the transfer function (2.2-6) if and only if $c^T = [1\ 0 \cdots 0]$.

Proof: The I-O Delay realization and the above system each produce the following N-stage input weighting sequences, respectively:

$$\begin{bmatrix} p^T \tilde{G} \\ p^T \tilde{F}\tilde{G} \\ \vdots \\ p^T \tilde{F}^{N-1}\tilde{G} \end{bmatrix} = [F^N \mid \hat{F}(N)]\,\tilde{G} = F^N G_1 + \hat{F}(N)\, G_2 = h(N)$$

$$\begin{bmatrix} c^T h(N) \\ c^T F\, h(N) \\ \vdots \\ c^T F^{N-1}\, h(N) \end{bmatrix} = \begin{bmatrix} c^T \\ c^T F \\ \vdots \\ c^T F^{N-1} \end{bmatrix} h(N) = \bar{R}_o\, h(N)$$

From Lemma 2.3.2, the input weighting sequences are equal since both systems realize the same transfer function. Thus c^T must satisfy the equality:

$$\bar{R}_o = I_N^o$$

Using the definition for F^k in Appendix A.2, we see that the equality holds only if $c^T = [1\ 0 \cdots 0]$. If $c^T = [1\ 0 \cdots 0]$, $\bar{R}_o = I_N^o$, so that the input

weighting sequences for the two system realizations are equal. Again by Lemma 2.3.2, these two realizations must realize the same transfer function, namely (2.2-6). Further, since $\overline{R}_o = I_N^o$ is nonsingular, and is the observability matrix for (2.3-6, 2.3-7), that system is completely observable. ■

Lemma 2.3.4

The system (2.3-6, 2.3-7) with $c^T = [1\ 0\ \cdots\ 0]$, which realizes the $N^{\underline{th}}$ order transfer function (2.2-6), is completely controllable if and only if N is the minimal system order; i.e., there are no pole-zero cancellations in (2.2-6).

Proof: Since the system is already observable, from Lemma 2.3.3, controllability implies the realization is minimal [22], which implies no further possible reduction in the order of (2.2-6). A minimal realization is always completely controllable [22]. ■

Having stated these lemmas we are now prepared to state and prove the main results of this section.

Proposition 2.3.1

The 2N-dimensional I-O Delay state realization (2.2-8)-(2.2-12) for the transfer function (2.2-6) is completely controllable if and only if N is the minimal system order; i.e., there are no pole-zero cancellations in (2.2-6). Thus, controllability depends on the parameter vector $\overline{p}$ only to the extent that the location of $\overline{p} \in R^{2N+1}$ does not introduce common roots between the numerator and denominator of (2.2-6).

Proof: The I-0 Delay state realization is completely controllable if and only if (2.3-3) is a nonsingular matrix [3]. Since the rank of a matrix is unaffected by reordering its columns, we rewrite (2.3-3) as:

$$R_c = \left[\begin{array}{cccc|cccc} F^{N-1}\, h(N) & \cdots & F\, h(N) & h(N) & h(N-1) & \cdots & h(1) & h(0) \\ \hline & 0 & & & & & I_N^o & \end{array}\right]$$

Using Lemma 2.3.1, $\det(R_c) = \det([F^{N-1}\, h(N) \cdots F\, h(N)\; h(N)])$. But nonsingularity of R_c is now reduced to nonsingularity of the controllability matrix for the system of Lemma 2.3.4. Therefore R_c is nonsingular if and only if N is the minimal system order. ∎

Proposition 2.3.2

The I-0 Delay state realization for the transfer function (2.2-6) is not completely observable, independent of the parameter vector $\bar{p}$.

Proof: The I-0 Delay state realization is completely observable if and only if (2.3-5) is a nonsingular matrix [3]. But R_o is clearly singular since the lower half of the matrix is a linear combination of the upper half:

$$[F^{2N} \,|\, F^N \hat{F}(N)] = F^N [F^N \,|\, \hat{F}(N)]$$ ∎

Given the necessary and sufficient conditions for controllability of the I-0 Delay state realization in Proposition 2.3.1, consider the following situation which may occur in the M-subsystem of the MRAS. Let $\bar{p}_M$ be such that the corresponding transfer function

$$H_M(z) = b_{M_o} + \frac{\sum_{i=1}^{N} (b_{M_i} + a_{M_i} b_{M_o}) z^{N-i}}{z^N - \sum_{i=1}^{N} a_{M_i} z^{N-i}} \tag{2.3-7}$$

has at least one pole and zero in common, giving subsystem M (in effect) a structure of lesser order (by at least 1) than that of the adaptable subsystem S. We introduce the notation $\mathcal{O}(M) < \mathcal{O}(S)$ to indicate this condition. For MRAS identification, this situation might correspond to an identification model with a higher order than that of the plant to be identified; for MRAS control, this might correspond to a reference model with a lower order than that of the plant/controller subsystem S. The MRAS control case is of particular interest. If we consider a MRAS structure where the reference model is characterized by an I-O delay difference equation of the form (2.2-1) with $2N_M+1$ parameters, while the plant is characterized by $2N_S+1$ parameters, $N_S > N_M$, then it is not clear that the Landau MRAS would still retain its hyperstable property. Thus the pole-zero cancellation strategy in defining a lower order reference model permits us to retain the hyperstable property of the MRAS.

We investigate the effect of M-subsystem pole-zero cancellation on the movement of the "excess" adaptable poles and zeroes in subsystem S, through simulation examples. Refer first to Example 2.2-2, case c. That 2nd order problem, without pole-zero cancellation, was successful in converging both state-error and parameter-error to zero. Now consider the same problem, except where the zero is shifted to cancel first the pole at $z = 0.9$, then the pole at $z = 0.4$.

Example 2.3-1

The problem is set up exactly as in Example 2.2-2, case (c), except that the sine input is (amplitude, period) = (1,20) + (4,3), and the M-subsystem zero is chosen to cancel one of the poles. The two cases considered here are:

case(a): M-subsystem poles at $z = (0.9, 0.4)$

zero at $z = 0.9$

$$\text{Thus } \bar{p}_M^T = [-0.36,\ 1.3 \mid -0.54,\ 0.6,\ 0]$$

case(b): M-subsystem poles at $z = (0.9, 0.4)$

zero at $z = (0.4)$

$$\text{Thus } \bar{p}_M^T = [-0.36,\ 1.3 \mid -0.04,\ 0.1,\ 0]$$

The simulation results are as follows, also depicted in Figure 2.3-1.

Final S-subsystem:

case	$\bar{p}^T(10)$	(poles, zero)
a)	$[0.207\ 0.35 \mid 0.028,\ 0.602,\ 8 \times 10^{-4}]$	$(-0.0517,\ 0.4 \mid -0.0465)$
b)	$[0.512,\ 0.331 \mid 0.056,\ 0.1,\ -3.4 \times 10^{-4}]$	$(0.9,\ -0.569 \mid -0.559)$

In both cases one S-subsystem pole converges to the uncancelled M-subsystem pole; the other S-subsystem pole and the zero tend to a common value, although not the pole-zero value chosen in the M-subsystem. Thus both subsystems' outputs appear as if from 1st order systems. The final common pole-zero value appears to be influenced by the choice of input. Further, we can show through simulation that a single input frequency is sufficient to cause the same type of parameter convergence property as has been demonstrated here. Notice that, in Example 2.2-2, case (b) with 5 parameters, a single input frequency was insufficient for parameter convergence, although it was sufficient for the 1st order problem in Example 2.2-1, case (b), with 3 parameters. This fact reinforces the contention that an M-subsystem pole-zero

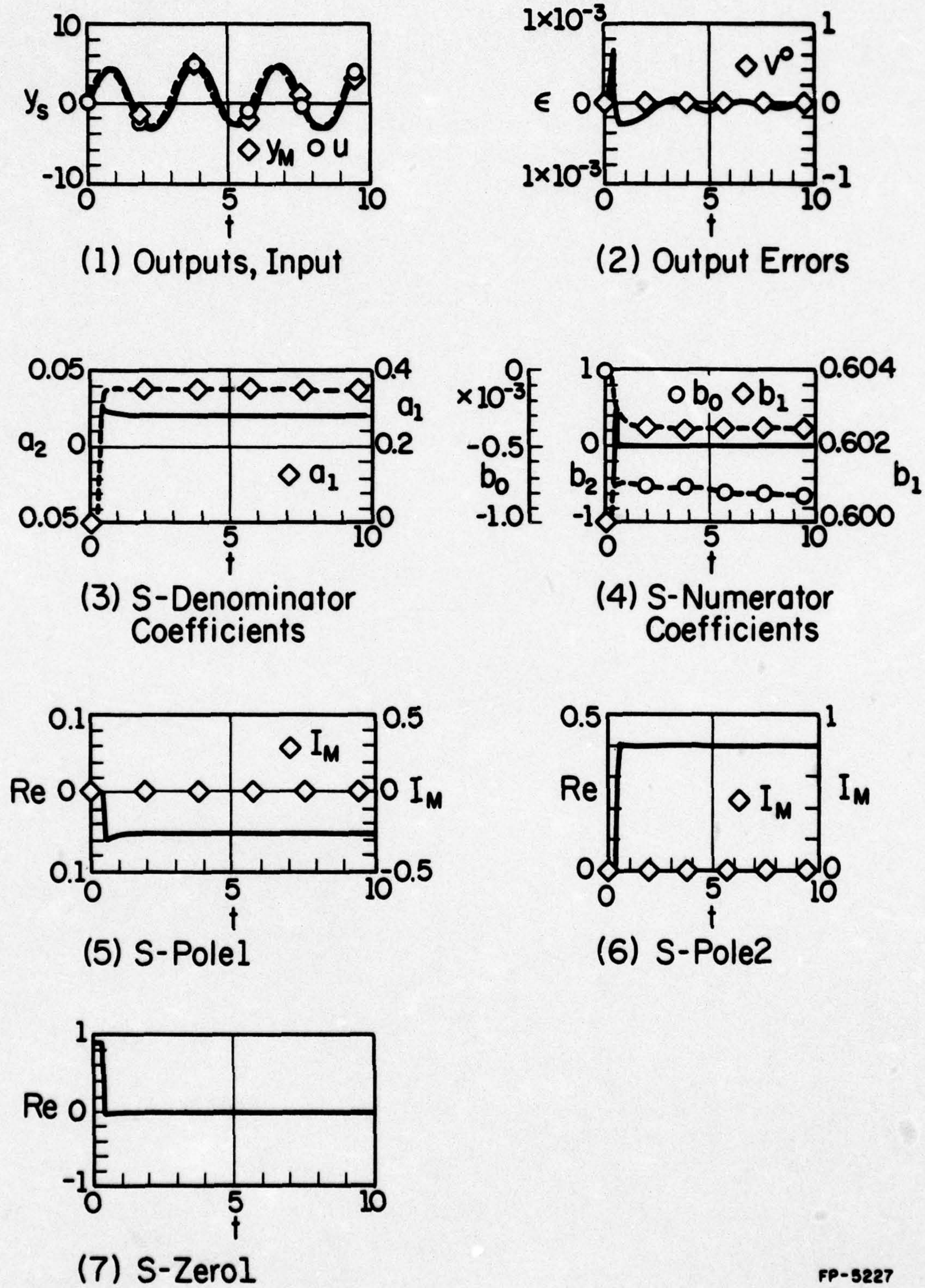

Figure 2.3-1(a) ELMA Example 2.3-1--Case a

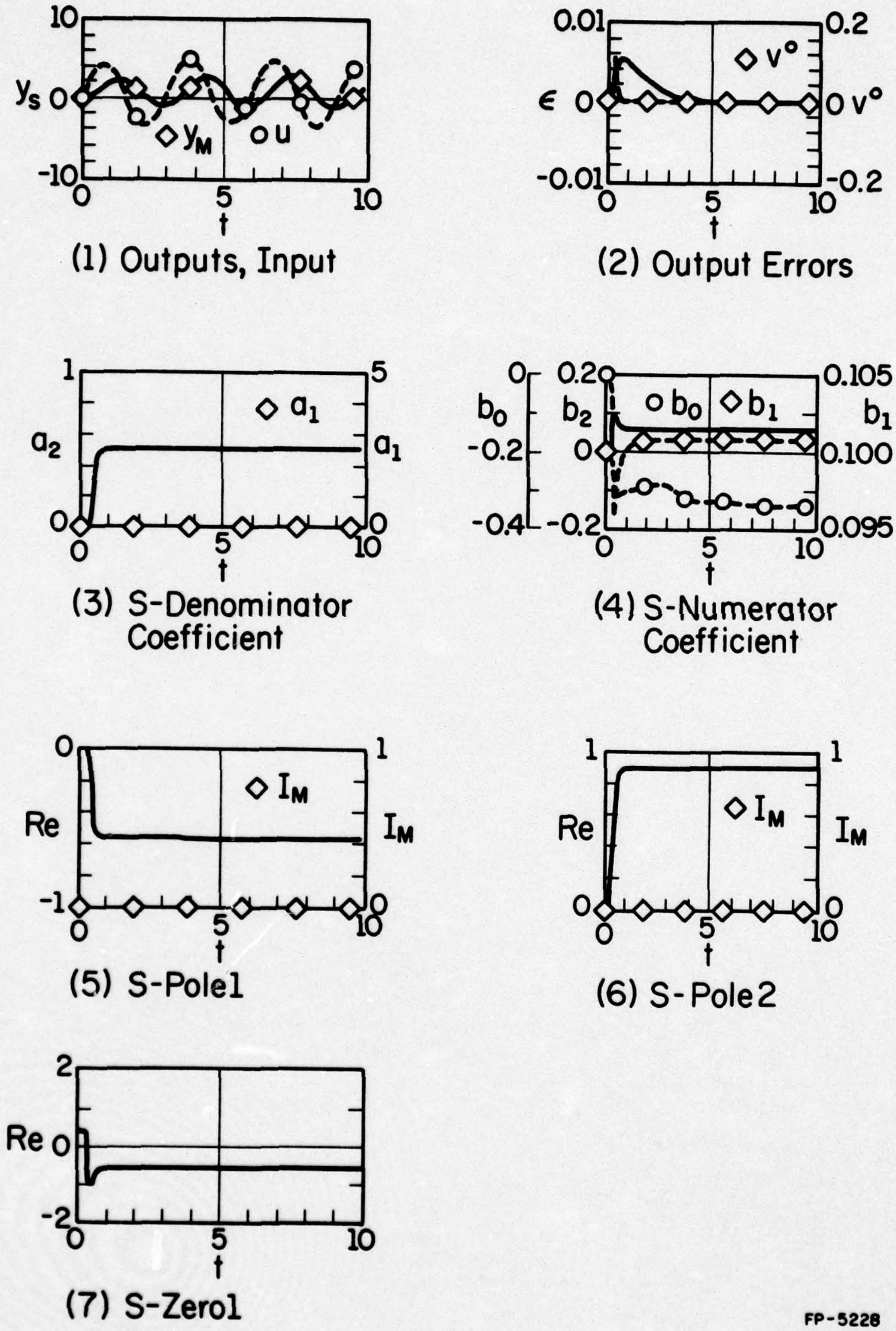

Figure 2.3-1(b) ELMA Example 2.3-1--Case b

cancellation is recognized by the adaptation process, which then solves a 1st order problem, while causing an arbitrary S-subsystem pole-zero cancellation. ■

In concluding the discussion on this point we consider one more aspect of pole-zero cancellation; in this case how it might affect the hyperstability properties of the MRAS, as established by Landau. Popov [50] assumes the linear subsystem of Figure 2.1-3 to be non-degenerate; that is, in the MRAS context, the state error e(k) is assumed completely controllable through B_M and completely observable through C_M^T, where (C_M^T, F_M, B_M) corresponds to the phase-canonic realization for (LS), as defined in (2.1-17) - (2.1-19). We note again that (LS) is a hyperstable subsystem if and only if $c \in R^N$, in (2.1-17), is chosen to guarantee (LS) has a positive real transfer function (2.1-20). However, one further assumption which must be made in choosing c, in order for (LS) to satisfy Popov's non-degeneracy assumption, is that the parameter vector for (LS):

$$\bar{p}_{LS}^T = [a_M^T \,\vdots\, c^T \,\vdots\, 1] \tag{2.3-8}$$

provide no pole-zero cancellations in (2.1-21). Otherwise (LS) will be degenerate. Landau does not consider this aspect of selecting the vector c in his MRAS design.

We turn now to the consideration of obtaining a linear algebraic relation between the 2N-dimensional I-0 Delay state and an arbitrary N-dimensional state, each of which realizes the same $N^{\underline{th}}$ order transfer function.

<u>Proposition 2.3.3</u>

Given an arbitrary N-dimensional realization for the transfer function (2.2-6):

$$w(k+1) = A\, w(k) + B\, u(k) \tag{2.3-9}$$

$$y(k) = c^T w(k) + b_o\, u(k) \tag{2.3-10}$$

and the I-0 Delay state realization (2.2.8 - 2.2-12), the state variables w(k) and x(k) are related by

$$R_o\, w(k) = \begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{N-1} \end{bmatrix} w(k) = [F^N \mid \hat{F}(N)]\, x(k) \tag{2.3-11}$$

where the matrices F^N, $\hat{F}(N)$ are defined in Appendix A.2.

<u>Proof</u>: Since both realizations represent the same transfer function, the zero-state output responses to an arbitrary input must be the same for both. Construct

the N-stage output sequence for each realization, $Y^T(N) = \{y(o), y(1), \cdots y(N-1\}$.

$$Y(N) = \begin{bmatrix} c \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix} w(o) + \begin{bmatrix} b_o & o & \cdots & o \\ CB & b_o & & \\ CAB & CB & b_o & o \\ \vdots & & & \\ CA^{N-2}B & & \cdots & b_o \end{bmatrix} \begin{bmatrix} u(o) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}$$

$$= \left[\begin{array}{c|l} a^T & b^T \\ a^T F & b^T I_N^1 \;\; + a^T \hat{F}(1) \\ \vdots & \\ a^T F^{N-1} & b^T I_N^{N-1} + a^T \hat{F}(N-1) \end{array}\right] \begin{bmatrix} x_1(o) \\ x_2(o) \end{bmatrix}$$

$$+ \begin{bmatrix} b_o & o & o \cdots & o \\ p^T \tilde{G} & b_o & o & \\ p^T \tilde{F} \tilde{G} & p^T \tilde{G} & b_o & \\ \vdots & & & o \\ p^T \tilde{F}^{N-2} \tilde{G} & & \cdots & b_o \end{bmatrix} \begin{bmatrix} u(o) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}$$

According to Lemma 2.3.2, the terms involving input must be equal. Using the definitions of F^N and $\hat{F}(N)$ in Appendix A.2, we obtain (2.3-11). ■

Two specific cases of N-dimensional SISO system realization for (2.2-6) are the completely controllable (phase) canonic and completely observable canonic forms [3]. In these two cases, (2.3-9, 2.3-10) become:

$$w_c(k+1) + F\, w_c(k) + G_2\, u(k) \tag{2.3-12}$$

$$y(k) = (b^T + b_o a^T)\, w_c(k) + b_o\, u(k) \tag{2.3-13}$$

$$w_o(k+1) = (P_N^o\, F^T\, P_N^o)\, w_o(k) + P_N^o(b + b_o a)\, u(k) \tag{2.3-14}$$

$$y(k) = G_2^T\, P_N^o\, w_w(k) + b_o u(k) \tag{2.3-15}$$

respectively, where a, b, b_o, F, G_2 are defined in (2.2-10 - 2.2-12), and P_N^o is defined in Appendix A.1.

Corollary 2.3.1

The completely controllable and completely observable states, $w_c(k)$ and $w_o(k)$, respectively are related to the I-0 Delay state by:

$$\left\{ \begin{bmatrix} b^T \\ b^T F \\ \vdots \\ b^T F^{N-1} \end{bmatrix} + b_o F^N \right\} w_c(k) = [F^N \mid \hat{F}(N)]\, x(k) \tag{2.3-16}$$

$$\begin{bmatrix} G_2^T \\ G_2^T F^T \\ \vdots \\ G^T F^{T(N-1)} \end{bmatrix} P_N^o\, w_o(k) = [F^N \mid \hat{F}(N)]\, x(k) \tag{2.3-17}$$

Proof: This follows from Proposition 2.3-3 by direct substitution, and recognizing from Appendix A the definition of F^N and that $P_N^o\, P_N^o = I_N^o$. ■

We next observe the following:

Proposition 2.3.4

Given the I-O delay state realization $x(k)$ and an arbitrary N-dimensional state realization $w(k)$ of (2.2-6), there does not exist $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ such that

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} w(k).$$

Proof: It is sufficient to consider only $x_2(k) = R_2 w(k)$. $x_2^T(k) = [u(k-N) \ldots u(k-1)]$, and clearly, $w(k)$ above is not sufficient to determine the past N inputs. The inputs necessary to reach $w(k)$ depend also on the starting point $w(k-N)$. ∎

It is possible to express $x_1(k)$ as a linear function of $w(k-N)$ and $x_2(k)$. Given the N-dimensional realization (C^T,A,B), we can show that:

$$x_1(k) = \begin{bmatrix} C^T \\ C^TA \\ \vdots \\ C^TA^{N-1} \end{bmatrix} w(k+N) + \begin{bmatrix} b_o & o & & o \\ C^TB & b_o & o & o \\ C^TAB & C^TB & b_o & \\ \vdots & & & o \\ C^TA^{N-2}B & B & & b_o \end{bmatrix} x_2(k) \quad (2.3\text{-}18)$$

However, no advantage in reducing the amount of ELMA computation is attained by using $w(k)$ instead of $x_1(k)$.

2.4 Parameter-Error Equilibria in MRAS Identification

Landau's MRAS algorithm, as well as others designed from Liapunov stability theory, guarantee only that the parameter-error, $\delta\overline{p}(k)$, between the M- and S-subsystems is bounded, not asymptotically stable, with respect to the origin of R^{2N+1}. In this section we examine in more detail the character of the trajectory $\delta\overline{p}(k)$, particularly its admissible equilibrium region. What constraints may we develop which characterize this region? Examination of these questions provides further insight into the stability properties of the Landau MRAS or the ELMA, as well as into how one might augment the ELMA design (through appropriate choice of input signal) to extend the property of asymptotic stability to the parameter-error. A fundamental issue in the study of parameter identification is that of identifiability. If the MRAS identification model contains more parameters than are necessary to characterize the system being identified, then it is not possible for the MRAS parameter-error to be guaranteed asymptotically stable. This property was illustrated in the last section in Example 2.3-1 where the identification model contained five parameters while the plant was capable of being completely characterized by only 3 parameters. In the remainder of this section we will assume identifiability conditions are satisfied.

We first survey published results dealing with asymptotic stability of the MRAS parameter-error. The problem has received detailed attention from several authors, and several different sufficient condition results on the input signal have been obtained which guarantee the asymptotic stability. Then we examine the implications on $\delta\overline{p}(k)$ of state-error convergence, which leads to a development of several geometric and algebraic representations

for the $\delta\bar{p}$-equilibrium subspace. Then we consider the problem of guaranteeing $\delta\bar{p}(k)$ asymptotically stable, taking a viewpoint based on the $\delta\bar{p}$-equilibrium subspace representation. Finally, we use simulation examples to substantiate that the sufficient conditions proposed by other authors are conservative.

In considering MRAS identification using stability-based design methods, most authors mention the problem of guaranteeing parameter-error convergence to the origin. Lion [40] appears to be the first author to confront this aspect of MRAS design in detail in the literature. He provides sufficient conditions to guarantee asymptotically stability of parameter-error. For the continuous-time MRAS identification problem posed by Lion, convergence of state-error implies an orthogonality condition of the form:

$$H(t) = x_M^T(t)\ \delta p^* = 0 \tag{2.4-1}$$

where δp^* denotes the final parameter-error, and $\bar{x}_M^T = [y_{M_o}, y_{M_1}, \ldots, y_{M_{n-1}}, u_o, u_1, \ldots, u_m]$ corresponds to a phase-variable state-representation obtained from "state-variable" filtering of the plant output and input, $y_M(t)$ and $u(t)$, respectively. Since $x_M(t)$ is time-varying, δp^* is constrained to the subspace E^* obtained from the intersection of the continuum of hyperspaces $\{H(t) \mid t \epsilon [t^*, \infty)\}$, there t^* denotes any time after

which e(t) has reached the origin; i.e.,

$$E^* = \bigcap_{t \in [t^*,\infty)} H(t) \qquad (2.4\text{-}2)$$

Lion's strategy, then, is to choose the MRAS input u(t) in such a way that the only possible solution for (2.4-2) is $E^* = \{o\}$; that is, $\delta p^* = o$. Sufficient conditions given by Lion to guarantee $\delta \overline{p}^* = o$ are:

1) u(t) is a sinusoid with at least $q^* = \frac{(n+m+1)}{2}$ distinct frequencies $\{w_1, \cdots, w_{q^*}\}$
2) The phase shift $\varphi(w)$ of y(t) in response to each w_i, $i = 1,2,\cdots, q^*$, must satisfy $\varphi(w_i) \neq k\pi$, $k=o, \pm 1, \pm 2, \cdots$

Kudva and Narendra [29,30] start with a frequency domain viewpoint for a discrete-time multi-input MRAS. They obtain rank conditions for a matrix constructed from the input sequence which, if satisfied, are sufficient to guarantee asymptotic stability of the parameter-error. This result differs from other approaches in that it neither requires an explicit frequency content condition on the input u(k), nor assumes a particular input class such as sinusoids. However, how the matrix to which a rank test is applied is obtained is not clear in their paper.

The most recent result, and an advancement over Lion, is reported by Kim and Lindorff [24]. By considering an arbitrary state-variable representation of a single-input linear plant model, they show that the matrix error $[\delta A(t) \;\; \delta B(t)]$ converges to o if:

1) $u(t)$ is periodic with at least q^* distinct frequencies $\{w_1, \cdots, w_{q^*}\}$, where

$$q^* = \min_{\text{integers}} \{q \mid q \geq R/2\}$$

$$R = \max_{i = 1, \cdots, n} \{R_i\}$$

R_i = number of non-zero elements in the $i^{\underline{th}}$ row of $[\delta A(t) \quad \delta B(t)]$

2) For at least one state variable, the phase shift $\varphi_j(w)$ satisfies:

$$\varphi_j(w_i) \neq \frac{k\pi}{2} \qquad \text{for all } i=1, \cdots, q^*; \quad k=0, \pm 1, \pm 2, \cdots$$

3) Given the set of non-zero columns of $\delta A(0)$, the corresponding set of plant state variable $\{x_{M_j}\}$ must be linearly independent in the steady-state.

When the state realization has minimal dimension = n, $q^* \leq \frac{n + 1}{2}$, with strict inequality often the case. This compares favorably with Lion's condition $q^* = \frac{n + m + 1}{2}$. The authors also extend this result to multi-input systems.

We will now use the ELMA of Section 2.2 to analyze some properties of the parameter-error $\delta\bar{p}(k)$. We first consider the implications of state-error convergence and $v^o(k)$ convergence on the adaptation process, concluding that $\delta\bar{p}(k)$ must reach a fixed point $\delta\bar{p}^*$ in either case. Then we develop several alternate geometric interpretations which characterize the subspace in which $\delta\bar{p}^*$ must necessarily be contained.

Consider the ELMA under the condition that state-error convergence has occurred; that is, $\varepsilon(k) = 0$, $e(k) = 0$. From (2.1-12, 2.1-14) we conclude that:

$$v(k) = v^o(k) = 0 \tag{2.4-3}$$

$$\delta\overline{p}(k+1) = \delta\overline{p}(k) = \delta\overline{p}^* \tag{2.4-4}$$

Thus the parameter-error $\delta\overline{p}(k)$ necessarily becomes a stationary point $\delta\overline{p}^*$ when state-error convergence has occurred.

Making use of (2.1-11, 2.1-13) and taking $c^T = -a^T(k)$, $v^o(k)$ may also be expressed in the form:

$$v^o(k) = \overline{x}_M^T(k)\ \delta\overline{p}(k) \tag{2.4-5}$$

According to (2.4-3), $v^o(k)$ will have converged to the origin if the state-error has already done so. However, $v^o(k)$ may also converge to the origin prior to $e(k)$. We may show this by algebraic manipulation of already available equations. Using (2.4-5) and (2.1-14) we obtain a recursive equation for $v^o(k)$:

$$v^o(k+1) = -\frac{\overline{x}_M^T(k+1)F(k)\overline{x}_S(k)}{1+\overline{x}_S^T(k)F(k)\overline{x}_S(k)}\,v^o(k) + \overline{x}_M^T(k+1)\delta\overline{p}(k) \tag{2.4-6}$$

Using (2.2-18, 2.1-10, 2.1-11), we obtain a new recursive equation for $e(k)$:

$$e(k+1) = F_S(k)e(k) + G_2\,\frac{v^o(k)}{1+\overline{x}_S^T(k)F(k)\overline{x}_S(k)} \tag{2.4-7}$$

where

$$F_S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & 0 \\ 0 & \cdots & & 0 & 1 \\ a_N(k) & \cdots & & & a_1(k) \end{bmatrix} \tag{2.4-8}$$

is a submatrix in the S-subsystem matrix $F(k)$. Now suppose $v^o(k) = 0$, implying $\overline{x}_M^T(k)\ \delta\overline{p}(k) = 0$, and that $\overline{x}_M(k+\ell)$ remains orthogonal to $\delta\overline{p}(k)$ for

all $\ell \geq 0$. Then $v_o(k+\ell) = 0$, $\delta\bar{p}(k+\ell) = \delta\bar{p}(k) = \delta\bar{p}^*$, and (2.4-7) is unforced

$$e(k+1) = F_S^* \, e(k). \tag{2.4-9}$$

$e(k)$ will be asymptotically stable if the final minimal set of poles (corresponding to the eigenvalues of F_S^*) for the S-subsystem are stable, and will converge to the origin at a rate dominated by the slowest eigenvalue of F_S^*. It is interesting to consider whether it is possible for F_S^* to have unstable eigenvalues. From our analysis thus far in this section, nothing prohibits such an occurrence. We will study this question in more detail in Section 2.5.

So far we have shown that $\delta\bar{p}(k)$ is a fixed point $\delta\bar{p}^*$ under either of the two conditions:

1) $v^o(k)$ has converged to 0

2) $\varepsilon(k)$ and $e(k)$ have converged to 0;

and that (2) implies (1), but not the converse. We now develop two geometric interpretations for the constraint that must be satisfied by $\delta\bar{p}^*$. The first approach is based on (2.4-5), the second on a new dynamic equation for $\delta\bar{p}(k)$.

When the MRAS adaptation process has ceased ($v^o(k) = 0$ for all k), the final parameter-error $\delta\bar{p}^*$ is constrained to lie successively in elements of an infinite sequence of hyperspaces [43], $\{H(k+\ell)\}$ defined by:

$$H(k+\ell) = \{\delta\bar{p}^* \, \varepsilon \, R^{2N+1} \mid X_M^T(k+\ell)\delta\bar{p}^* = 0, \ \ell \, \varepsilon \, [0,\infty)\} \tag{2.4-10}$$

where k is any stage after which v^o has converged. Let $\{H^*(k+\ell)\}$ be a sequence of subspaces in R^{2N+1} defined by

$$\left.\begin{aligned} H^*(k) &= R^{2N+1} \\ H^*(k+\ell+1) &= H^*(k+\ell) \cap H(k+\ell) \end{aligned}\right\} \qquad (2.4\text{-}11)$$

where $\cap$ denotes the set intersection operator. Clearly, $H^*(k+\ell+1) \quad H^*(k+\ell)$. since each $H(k)$ is a closed set,

$$\overline{H}^* = \lim_{\ell \to \infty} H^*(k+\ell) \qquad (2.4\text{-}12)$$

is also a closed set [4]. Since $\delta\overline{p}^*$ lies in every $H(k+\ell)$, it also lies in every $H^*(k+\ell)$, and finally in $\overline{H}^*$. In fact, $\overline{H}^*$ completely characterizes the subspace in R^{2N+1} within which $\delta\overline{p}^*$ is constrained by the state vector sequence $\{\overline{x}_M(k+\ell)\}$.

This geometric viewpoint has an algebraic counterpart. Corresponding to each hyperspace $H(k+\ell)$ is a linear functional equation (a linear transformation from R^{2N+1} to R):

$$x_M^T(k+\ell)\ \delta\overline{p}^* = 0 \qquad (2.4\text{-}13)$$

which follows from (2.4-5). Consider the linear matrix equation:

$$X_M^T(\overline{\ell})\,\delta\overline{p}^* = 0 \qquad (2.4\text{-}14)$$

where

$$X_M^T(\overline{\ell}) = \begin{bmatrix} \overline{x}_M^T(k) \\ \overline{x}_M^T(k+1) \\ \vdots \\ \overline{x}_M^T(\overline{\ell}) \end{bmatrix} \qquad (2.4\text{-}15)$$

and $\bar{\ell}$ is the least integer $\ell \in [0,\infty)$ such that $\rho\{X_M(\bar{\ell})\} = \rho\{X_M(\ell)\}$ for all $\ell \geq \bar{\ell}$. In other words, adding more stages to $X_M(\bar{\ell})$ will not increase the rank of $X_M^T(\bar{\ell})$. The null-space of $X_M^T(\bar{\ell})$ is identical to the subspace $\bar{H}^*$ in (2.4-12):

$$\mathcal{N}\{X_M^T(\bar{\ell})\} = \bar{H}^* \tag{2.4-16}$$

We note that, if $X_M(\bar{\ell})$ is full rank, $\rho = 2N + 1$, then $\mathcal{N}\{X_M^T(\bar{\ell})\} = \bar{H}^* = \{0\}$, and (2.4-14) has the unique (trivial) solution $\delta\bar{p}^* = 0$. This means parameter identification has been successful.

The second approach to a geometric description of the constraint subspace for $\delta\bar{p}^*$ is based on the following autonomous time-varying linear equation for $\delta\bar{p}(k)$, derived from (2.4-5) and (2.1-14):

$$\delta\bar{p}(k+1) = P(k)\ \delta\bar{p}(k) \tag{2.4-17}$$

where

$$P(k) = I^o_{2N+1} - \frac{F(k)\bar{x}_S(k)\ \bar{x}_M^T(k)}{1 + \bar{x}_S^T(k)F(k)\bar{x}_S(k)} \tag{2.4-18}$$

When $v^o(k) = 0$ for all k, $\delta\bar{p}(k+1) \underset{=}{} \delta\bar{p}(k) = \delta\bar{p}^*$. The following proposition establishes the basis for the subspace in which $\delta\bar{p}^*$ is constrained.

<u>Proposition 2.4.1</u>

Matrix P(k) has an eigenvalue $\lambda = 1$ of multiplicity $m = 2N$, and full degeneracy $q = 2N$ if and only if $\bar{x}_S(k) \neq 0$.

<u>Proof</u>

$$\rho\{[\lambda I^{o}_{2N+1} - P(k)]_{\lambda=1}\} = \rho\{\frac{F(k)\bar{x}_S(k)\bar{x}_M^T(k)}{1 + \bar{x}_S^T(k)F(k)\bar{x}_S(k)}\} = 1$$

since $F(k) > 0$ and $\rho\{\bar{x}_S(k)\bar{x}_M^T(k)\} = 1$. Thus $\lambda = 1$ is an eigenvalue of $P(k)$ with degeneracy $q = (2N+1)-\rho = 2N$. It can be shown that $P(k)$ has one eigenvalue

$$\tilde{\lambda} = \frac{1 - \bar{x}_S^T(k)F(k)\bar{e}(k)}{1 + \bar{x}_S^T(k)F(k)\bar{x}_S(k)},$$

where $\bar{e}^T(k) = [e^T(k) \vdots o \ \ldots \ o]$. Only when $\bar{x}_S(k) = 0$ will $\tilde{\lambda} = 1$, giving $\lambda = 1$ a multiplicity $m = 2N + 1$. Otherwise, multiplicity for $\lambda = 1$ is $m = 2N$, and degeneracy is full, $q = m = 2N$. ∎

From Proposition 2.4.1 and (2.4-17), $\delta\bar{p}^*$ is an eigenvector for $P(k)$, and since $\lambda = 1$ has multiplicity $m = 2N$ and full degeneracy, 2N linearly independent eigenvectors for $\lambda = 1$ exist. Thus $\delta\bar{p}^*$ is constrained at stage k to lie in a (2N-dimensional) hyperspace, which we again denote $H(k)$:

$$H(k) = \text{span}\ \{z_1(k) \ \ldots \ z_{2N}(k)\} \qquad (2.4\text{-}19)$$

where $\{z_1(k) \ \ldots \ z_{2N}(k)\}$ is the set of eigenvectors for $P(k)$ associated with $\lambda = 1$. Thus $\delta\bar{p}^*$ is constrained successively to the sequence of hyperspaces $\{H(k+\ell)\,|\,\ell \geq 0\}$.

The two hyperspaces (2.4-10) and (2.4-20) are identical; we show this informally with the following argument:

$\dim \mathcal{N}\{\bar{x}_S\bar{x}_M^T\} = 2N$, thus $\mathcal{N}\{\bar{x}_S\bar{x}_M^T\}$ corresponds to a hyperspace in R^{2N+1}. Now, (2.4-17) implies $(\bar{x}_S\bar{x}_M^T)\delta\bar{p}^* = 0$, which in turn implies $\bar{x}_S(\bar{x}_M^T\delta\bar{p}^*) = 0$. Therefore, $\bar{x}_M^T\delta\bar{p}^* = 0$, which is simply (2.4-5), the basis for (2.4-10).

We now use the algebraic characterization for the $\delta\bar{p}^*$ constraint subspace to examine conditions on the input sequence for $\delta\bar{p}^*$ to be constrained to the origin. We stated earlier that (2.4-14) has the unique solution $\delta\bar{p}^* = 0$ if and only if $X_M(\bar{\ell})$ is full rank, $\rho = 2N + 1$. In order to examine when $X_M(\bar{\ell})$ is full rank we substitute for each $\bar{X}_M(k+\ell)$ the equivalent expression using $\bar{X}_M(k)$ and $\{u(k+1),\ldots u(k+\ell)\}$. Then we group terms into four submatrices, obtaining:

$$X_M(\bar{\ell}) = \left[\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right], \tag{2.4-20}$$

the submatrices X_{11}, X_{12}, X_{21}, X_{22} having dimensions $N \times (N+1)$, $N \times (\bar{\ell}-N)$, $(N+1) \times (N+1)$, and $(N+1) \times (\bar{\ell}-N)$, respectively,

where

$$X_{11} = \begin{bmatrix} [[I_N^o \vdots 0]x_M(k) \,|\, [F_M \vdots \hat{F}_M(1)]x_M(k) \vdots \cdots \vdots [F_M^N \vdots \hat{F}_M(N)]x_M(k)] \\ + [h_M(N-1) \; \ldots \; h_M(0)] \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & u(k) \\ \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & 0 & & & \\ \vdots & 0 & u(k) & & & \\ 0 & u(k) & u(k+1) & \cdots & & u(k+N-1) \end{bmatrix} \end{bmatrix} \tag{2.4-21}$$

$$
X_{12} = \left[\begin{array}{l}
[F_M [F_M^N \vdots \hat{F}_M(N)] x_M(k) \vdots \; \cdots \; \vdots F_M^{\bar{\ell}-N} [F_M^N \vdots \hat{F}_M(N)] x_M(k)] \; + \\[2ex]
[F_M^{\bar{\ell}-(N+1)} h_M(N) \vdots \; \cdots \; \vdots F h_M(N) \vdots h_m(n) \vdots \; \cdots \; \vdots h_M(0)] \\[2ex]
\qquad\qquad\qquad\qquad\qquad \begin{bmatrix}
0 & 0 & \cdots\; 0 & u(k) \\
\vdots & 0 & & \\
 & u(k) & & \\
u(k) & u(k+1) & \cdots & u(k+\bar{\ell}-(N+1)) \\
u(k+1) & & \cdots & u(k+\bar{\ell}-N) \\
\vdots & & & \\
u(k+N) & & \cdots & u(k+\bar{\ell}-1)
\end{bmatrix}
\end{array} \right]
\tag{2.4-22}
$$

$$
X_{21} = \begin{bmatrix}
u(k-N) & u(k-(N-1)) \cdots & u(k) \\
 & \vdots & u(k+1) \\
\vdots & & \vdots \\
 & u(k) & \\
u(k) & u(k+1) \quad \cdots & u(k+N)
\end{bmatrix}
\tag{2.4-23}
$$

$$
X_{22} = \begin{bmatrix}
u(k+1) & \cdots & u(k+\bar{\ell}-N) \\
\vdots & & \\
u(k+N+1) & \cdots & u(k+\bar{\ell})
\end{bmatrix}
\tag{2.4-24}
$$

where F_M, $\hat{F}_M(\cdot)$ and $h_M(\cdot)$ are defined in Appendix A and (2.3-4).

In order for (2.4-21) to have rank $\rho = 2N + 1$, it is necessary and sufficient that we be able to reorder columns so the first $2N + 1$ reordered columns forms a non-singular submatrix. However, it is not clear what

necessary and sufficient conditions the input sequence $\{u(k), \ldots, u(k+\bar{\ell})\}$ must satisfy in order for this general reordering to be possible.

We consider instead sufficient conditions for (2.4-21) to have full rank. From these sufficient conditions we are able to make some partial conclusions about the input sequence. The sufficient conditions are stated in the following Lemma:

<u>Lemma 2.4.1</u>

The matrix $X_M(\bar{\ell})$ is full rank, $\rho = 2N + 1$, if:

1) X_{21} is nonsingular

2) $\begin{bmatrix} X_{12} \\ X_{22} \end{bmatrix}$ has rank $\rho \geq N$

3) There exist N linearly independent columns in $\begin{bmatrix} X_{12} \\ X_{22} \end{bmatrix}$ such that they span the N-dimensional subspace of R^{2N+1} which is complementary to the (N+1)-dimensional subspace spanned by $\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix}$ ■

Condition (1) implies the input sequence $\{u(k-N), \ldots, u(k+N)\}$ must be such that (2.4-24) is nonsingular. Viewing $\{u(k-N), \ldots, u(k+N)\}$ as a vector element $U_N \in R^{2N+1}$, this vector "almost always" yields a nonsingular X_{21}, except when U_N lies on a certain nonlinear hypersurface defined by det $(X_{21}) = 0$. For example, for N = 1 and N = 2, respectively:

$$\det (X_{21}) = u(k-1)\, u(k+1) - u^2(k)$$

$$\det (X_{21}) = u(k-2)u(k)u(k+2) + 2u(k-1)u(k)u(k+1)$$
$$- u^3(k) - u(k-2)u^2(k+1) - u^2(k-1)u(k+2)$$

Conditions (2) and (3) provide little insight into properties which should be satisfied by the input sequence $\{u(k), \ldots, u(k+\bar{\ell})\}$. However, we make the following remark. In an effort to find conditions which move in the direction of being both necessary and sufficient to guarantee $X_M(\bar{\ell})$ full rank, Lemma 2.4.1 suggests that such conditions will depend both on the input sequence and on the M-subsystem parameter vector $\bar{p}_M$. The latter dependence follows from definitions for F_M, $\hat{F}_M(\cdot)$, and $h_M(\cdot)$. Thus, necessary and sufficient conditions on the input sequence to guarantee $X_M(\bar{\ell})$ full rank must be expressed in terms of $\bar{p}_M$, and would be dependent upon the specific plant being identified. This is, perhaps, why all previous investigations into this problem have focused on sufficient conditions only, at a level which depends only on the order of the MRAS, or number of parameters, but independent of any plant parameter values.

We conclude this section with further consideration of the conditions proposed by Lion and by Kim-Lindorff, described earlier, which deal with the frequency content of periodic inputs sufficient to guarantee parameter-error convergence to 0. When the Lion and Kim-Lindorff conditions are each applied to the ELMA, the same sufficient frequency content limit q* is obtained in each case. In Lion's case,

$$q^* \geq \frac{2N+1}{2} = N + 1$$

In the Kim-Lindorff case, we consider the I-O Delay state variable representation (2.2-8)-(2.2-12), and conclude that R = 2N + 1 from the Nth row of $[\delta\tilde{F} \vdots \delta\tilde{G}]$. Thus,

$$q^* = \min \{q | q \geq R/2\} = N + 1$$

From simulation experience, however, (see Examples 2.2-1 and 2.2-2) it appears that $q^* = N$ may always be sufficient for parameter-error convergence. We consider now two more examples to substantiate this conjecture, first for $N = 3$, and then for $N = 4$.

Example 2.4-1, $N = 3$

M-subsystem:

a) 3 poles at $z = (0.9, 0.4, -0.5)$

b) 1 zero at $z = (0.8)$

c) $b_o = 0$

d) b_2 chosen to provide unity gain on $|z| = 1$.

Thus $\overline{p}_M^T = [-0.18, 0.29, 0.8 \vdots -0.36, 0.45, 0, 0]$

Initial S-subsystem:

a) poles at $z = (0, 0, 0)$

b) S-transfer function numerator = M-transfer function numerator.

Thus $\overline{p}^T(0) = [0, 0, 0 \vdots -0.36, 0.45, 0, 0]$

A-subsystem:

a) $F(0) = \text{diag}\,[1000, \ldots, 1000]$

b) $c(t) = -a(t)$

Input $u(k)$:

sine with (amplitude, period) = (1,20) + (3,7) + (8,4)

Final S-subsystem:

$\bar{p}^T(10) = [-0.18, 0.291, 0.799 \vdots -0.361, 0.451, 10^{-5}, 10^{-5}]$,

corresponding to (poles, zeroes) = (0.9, 0.4, -0.501 $\vdots$ 0.8, 38500)

Simulation results are shown in Figures 2.4-1. We use a sinusoidal input u(k) with N = 3 frequencies, and find that parameter identification is successful to within 10^{-3} after about 3 seconds (30 discrete stages). The figure also indicates that $\varepsilon(k)$ is converging to the origin. As was shown earlier in this section, after $v^o(k)$ has converged, the motion of $\varepsilon(k)$ is governed by the final poles of the S-subsystem. Since one pole is at z = 0.9, a relatively slow transient for $\varepsilon(k)$ should be anticipated. Parameter trajectories for (a_1, b_o) are not shown in the figure. ■

<u>Example 2.4-2</u> N = 4

M-subsystem:

a) 4 poles at z = (0.9, 0.4, -0.5, 0)

b) 1 zero at z = (0.8)

c) $b_o = 0$

d) b_3 chosen to provide unity gain on $|z| = 1$

Thus $\bar{p}_M^T = [0, -0.18, 0.29, 0.8 \vdots -0.36, 0.45, 0, 0, 0]$

Initial S-subsystem:

a) poles at z = (0, 0, 0, 0)

b) S-transfer function numerator = M-transfer function numerator.

Thus $\bar{p}^T(0) = [0, 0, 0, 0 \vdots -0.36, 0.45, 0, 0, 0]$

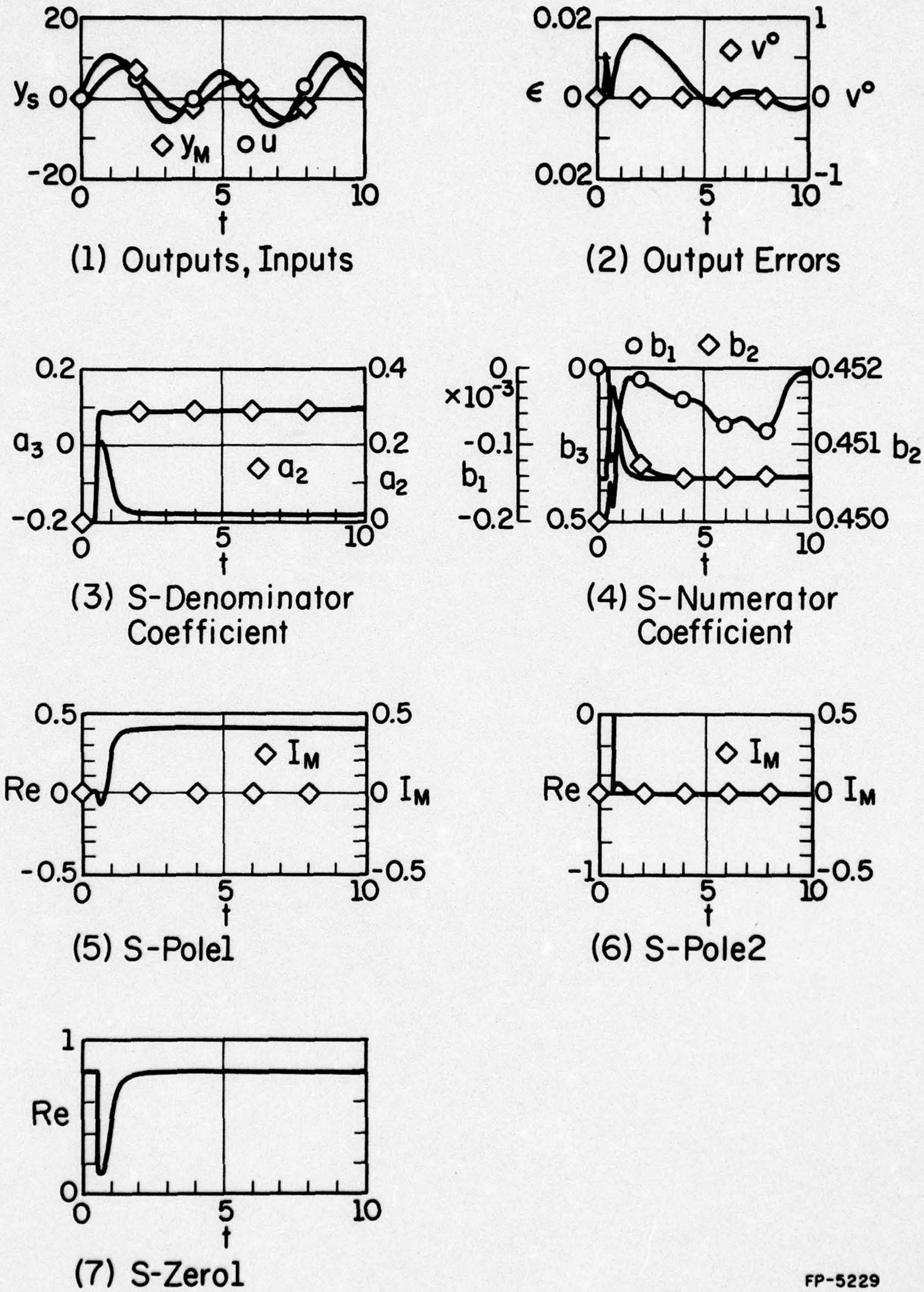

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Figure 2.4-1 ELMA Example 2.4-1

A-subsystem:

a) $F(0) = \text{diag}\ [1000, \ldots, 1000]$

b) $c(t) = -a(t)$

Input $u(k)$:

sine with (amplitude, period) = (1, 20) + (3,13) + (7, 7) + (12, 4).

Final S-subsystem:

$\bar{p}^T(10) = [-0.8 \times 10^{-3}, -0.179, 0.29, 0.8 \vdots -0.36, 0.45, 10^{-6}, -10^{-4}, 10^{-4}]$, corresponding to (poles, zeroes) = $(0.9, 0.402, -0.498, -0.46 \times 10^{-2} \vdots 0.8, 174., -174.)$

Simulation results are shown in Figure 2.4-2. We use a sinusoidal input with N = 4 frequencies, and find that parameter identification is successful to within 10^{-3} after about 3 seconds. Parameter trajectories for $(a_2, a_1, b_2, b_1, b_o)$ are not shown in the figure. ■

2.5 Analysis of Instability in the Landau MRAS Identification Algorithm

In this section we will examine the instability inherent in the Landau MRAS or ELMA identification algorithm. Although the algorithm is structured in accordance with Popov's hyperstability conditions, it is not generally possible to satisfy the Popov necessary and sufficient condition on the linear subsystem of Figure 2.1-3. A vector $c^T = [c_N \ldots c_1]$ must be chosen so that the linear subsystem transfer function

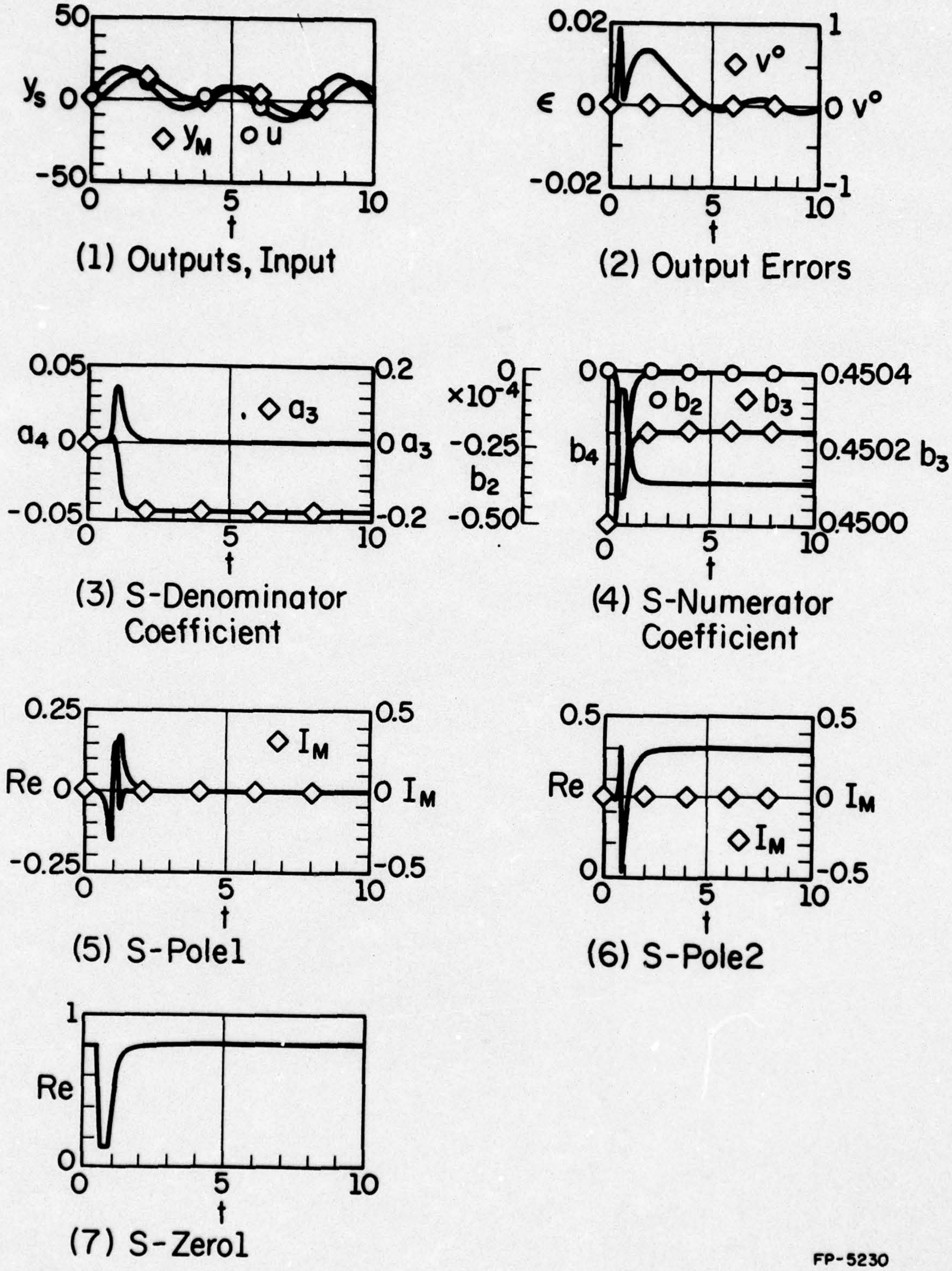


Figure 2.4-2 ELMA Example 2.4-2

$$H(z) = \frac{z^N + \sum_{i=1}^{N} c_i z^{N-i}}{z^N - \sum_{i=1}^{N} a_{M_i} z^{N-i}} = 1 + \frac{\sum_{i=1}^{N} (a_{M_i} + c_i) z^{N-i}}{z^N - \sum_{i=1}^{N} a_{M_i} z^{N-i}} \tag{2.5-1}$$

is positive-real. However, since the M-subsystem parameter-vector $a_M^T = [a_{M_N} \ldots a_{M_1}]$ is unknown, it is not possible to evaluate the positive-real property of (2.5-1), no matter what choice is made for c. A desirable choice for c would be

$$c = - a_M, \tag{2.5-2}$$

since, in this case, H(z) = 1, clearly positive-real. Landau suggests [35] a strategy where first one obtains an estimate for a_M, denoted $\hat{a}_M$, by a method with relatively low computational complexity, such as generalized least-squares [2], and then uses this estimate both for the initial identification model parameter value, a(0), and for c; that is, assign

$$c = - \hat{a}_M = - a(0). \tag{2.5-3}$$

A natural extension of this idea (and one adopted in the ELMA identification algorithm) is to specify that c track the value of a(k); that is,

$$c(k) = - a(k), \tag{2.5-4}$$

the strategy being for c(k) to approach $-a_M$ as the MRAS identification procedes to its successful conclusion. The success of this strategy will depend on the starting point a(0), and may present a problem particularly in those

practical situations where it is not feasible to obtain an adequate initial estimate for a_M.

Choosing c defined by (2.5-4) does not guarantee hyperstability for the MRAS. Thus we would like to find the region in R^n in which c(k) must remain in order to guarantee a hyperstable MRAS. This MRAS in the Popov configuration no longer conforms to the assumption that the linear subsystem is time-invariant. We might consider viewing (2.5-1) as a linear time-varying system, and then determine under what condition on c(k) (2.5-1) satisfies the time-varying version of the Discrete Positive Real Lemma [35]. Based on simulation results, we expect this region for $c(k) \in R^n$ is larger than the region constraining $c = \text{constant}$. For $c = \text{constant}$ it is required that, as part of the positive-real constraint on (2.5-1), the numerator polynomial of (2.5-1) have all its roots inside the unit circle. However, for the case (2.5-4) it has been observed that, during a finite number of stages of stable MRAS operation, the roots of the numerator polynomial may lie outside the unit circle.

We begin this section by discussion the difficulty of testing the positive-real condition for (2.5-1). Two possible approaches for determining whether a transfer function is positive-real are suggested. These methods are of use only for analysis, where we shall assume the plant parameter vector a_M is "known." Thus, these

tests would not be useful in an on-line identification context. We then direct our attention to the adaptation gain equation (2.1.15) to observe what effect MRAS instability has on F(k). Finally, we consider whether it is possible for the Landau MRAS identification problem to be restructured or reformulated in a way which retains hyperstability but avoids the potential instability problems related to the positive-real condition.

Definition 2.5.1 [16] A discrete-time transfer function $H(z) = q(z)/p(z)$, where q is order m, p is order n, is strictly positive-real if:

1) All roots of $p(z)$ and $q(z)$ lie strictly within the unit circle of the complex z-plane.
2) $H(e^{j\omega T}) + H(e^{-j\omega T}) > 0$ for all $\omega \geq 0$, where T is the fixed time interval between discrete stages.

Since $\mathrm{Re}[H(e^{j\omega T})]$ is an even function of ω, and $\mathrm{Im}[H(e^{j\omega T})]$ is odd, condition (2) may be expressed $\mathrm{Re}[H(e^{j\omega T})] > 0$. For example, when $N = 1$, the transfer function (2.5-1) becomes

$$H(z) = \frac{z + c_1}{z - a_{M_1}} = 1 + \frac{a_{M_1} + c_1}{z - a_{M_1}}, \tag{2.5-5}$$

and may be shown to be positive-real only if $|a_{M_1}| < 1$ and $|c_1| < 1$. In this simple case, the choice of c is independent of a_M. When $N \geq 2$, the analysis becomes more complicated, and the positive-real region for c, given a_M, will depend on a_M. A general set of inequalities, relating c to a_M, which guarantees (2.5-1) is positive-real is not known to have been derived. In fact, condition (2) applied directly generates a continuum of inequalities involving c and a_M, parametrized by ω. However, the continuum may be reduced to a

finite set when one recognizes the polynomial form of the inequalities. For example, when N = 2, condition (2) becomes

$$y(c,a_M,x) = 2(c_2-a_2)x^2 + [c_1(1-a_2)-a_1(1+c_2)]x$$
$$+ [1+(a_2-c_2)-(a_1c_1+a_2c_2)] > 0, \qquad (2.5\text{-}6)$$

a quadratic polynomial in $-1 \leq x = \cos \omega T \leq 1$. By the nature of the quadratic form on the interval [-1, 1], we can state conditions for the quadratic to be positive on this interval:

1) $y(c,a_M,+1) > 0$
2) $y(c,a_M,-1) > 0$
3) If $x_{min} \leq 1$, then $y_{min} = (c, a_M, x_{min}) > 0$, where x_{min} is the value of x that minimizes $y(c,a_M, \ )$ for fixed c, a_M.

Expressed in terms of c and a_M, and taking into account the stability constraint on a_M, these three conditions may be reduced to:

1') $1 + c_1 + c_2 > 0$

2') $1 - c_1 + c_2 > 0$

3') If $-1 \leq \dfrac{a_1(1+c_2) - c_1(1-a_2)}{4(c_2-a_2)} \leq 1,$

$$\text{then } \frac{-[c_1(1-a_2) - a_1(1+c_2)]^2}{8(c_2-a_2)} + [1+(a_2-c_2) - (a_1c_1+a_2c_2)] > 0$$

The first two conditions are seen to reduce to a a subset of the numerator, q(z), stability constraint inequalities. Even for this low order case, N = 2,

the inequality ($3'$) cannot be solved explicitly for c in terms of a_M. For arbitrary N, although the problem has a theoretical solution it is not a simple matter to select c, even when a_M is known.

The only feasible approach known to this author for analyzing the positive-real condition of a transfer function is to apply a test to specific numeric examples to determine whether the example violates conditions specified by the test. This approach would provide a trial-and-error method for examining the c-sensitivity question stated above, and would require that the plant parameters be assumed known during the analysis.

Two different tests can be proposed in this context. The first involves the possible application of the discrete positive-real lemma [16] to the given transfer function, requiring first that a state-variable realization for (2.5-1) be chosen. This lemma gives necessary and sufficient conditions for which the transfer function is strictly positive-real in terms of a set of matrices satisfying three matrix equations. One approach in applying this lemma would be to develop an iterative algorithm based on the three matrix equations which would converge to matrices satisfying the conditions of the lemma if the transfer function is positive-real, and would fail to converge otherwise. This author is not familiar with such an algorithm; the development of one could be the subject of future research in this area.

The second approach involves application of algebraic criteria obtained by Siljak [55] for positive-realness of discrete-time transfer functions. The Siljak test is attractive because it involves finding the locations of roots of two polynomials with respect to the unit circle. The test is formulated recursively, and although the second stage of the test is complicated, it is amenable to computation.

The MRAS exhibits unstable operation by the fact that the output error $\epsilon(k)$ becomes unbounded. This implies that $y_S(k)$ is unbounded, and thus the poles of the S-subsystem (determined by a(k)) must lie outside the unit circle. Once this unstable condition occurs in the MRAS, the adaptation gain F(k) responds in such a way that adaptation of the S-subsystem poles is effectively halted. We will analyze the adaptation gain dynamics with the goal of obtaining insight into a process in the MRAS algorithm which is linked with the unstable condition.

The adaptation gain equation for the MRAS identification algorithm is:

$$F(k+1) = F(k) - \frac{[F(k)\bar{x}_S(k)][F(k)\bar{x}_S(k)]^T}{1 + \bar{x}_S^T(k)F(k)\bar{x}_S(k)} \tag{2.5-7}$$

Assuming $F(0) = F^T(0) > 0$, it can be shown that

$$0 < F(k+1) \leq F(k), \tag{2.5-8}$$

that is, $F(k)$ remains positive definite for all k, independent of $\bar{x}_S(k)$, and approaches $F = 0$ in the limit as $k \to \infty$, independent of $\bar{x}_S(k)$, if $\bar{x}_S = 0$ for only a finite number of stages.

We shall partition $F(k)$ to correspond with $\bar{x}_S(k)$ as partitioned in (2.1-2). Thus

$$F(k) = \begin{matrix} F_1(k) & F_{12}(k) \\ F_{12}^T(k) & F_2(k) \end{matrix} \tag{2.5-9}$$

where F_1 is N x N, F_2 is (N+1) x (N+1), and the gain equation 2.5-7 may be decomposed to:

$$F_1(k+1) = F_1(k) - \frac{F_1 \mathring{x}_S \mathring{x}_S^T F_1 + F_1 \mathring{x}_S \bar{x}_S^{1\,T} F_{12} + F_{12} \bar{x}_S^{1} \mathring{x}_S^{T} F_1 + F_{12} \bar{x}_S^{1} \bar{x}_S^{1\,T} F_{12}^{T}}{1 + \bar{x}_S^T F \bar{x}_S} \tag{2.5-10a}$$

$$F_{12}(k+1) = F_{12}(k) - \frac{F_1 \mathring{x}_S \mathring{x}_S^T F_{12} + F_1 \mathring{x}_S \bar{x}_S^{1\,T} F_2 + F_{12} \bar{x}_S^{1} \mathring{x}_S^{T} F_{12} + F_{12} \bar{x}_S^{1} \bar{x}_S^{1\,T} F_2}{1 + \bar{x}_S^T F \bar{x}_S} \tag{2.5-10b}$$

$$F_2(k+1) = F_2(k) - \frac{F_{12}^T \mathring{x}_S \mathring{x}_S^T F_{12} + F_{12}^T \mathring{x}_S \bar{x}_S^{1\,T} F_2 + F_2 \bar{x}_S^{1} \mathring{x}_S^{T} F_{12} + F_2 \bar{x}_S^{1} \bar{x}_S^{1\,T} F_2}{1 + \bar{x}_S^T F \bar{x}_S} \tag{2.5-10c}$$

where the time-index k has been suppressed for convenience. From the parameter adaptation equation (2.1-14), and using the above partitioning of F(k), that part of the parameter adaptation affecting the S-subsystem poles (i.e., a(k)) may be written:

$$a(k+1) = a(k) + \frac{F_1(k)x_S^o(k) + F_{12}(k)\bar{x}_S^{-1}(k)}{1 + \bar{x}_S^T(k)F(k)\bar{x}_S(k)} \tag{2.5-11}$$

When instability occurs, $\| x_S^o(k) \|$, corresponding to the S-subsystem N-stage delayed output sequence, becomes unbounded, while $\| \bar{x}_S^{-1}(k) \|$, corresponding to the (N+1)-stage delayed input sequence, remains finite.

<u>Proposition 2.5.1</u>

In the limit as $\| x_S^o(k) \| \to \infty$, $\| \bar{x}_S^{-1}(k) \|$ bounded,

$$F(k+1) = \begin{bmatrix} 0 & 0 \\ & \\ 0 & F_2(k+1) \end{bmatrix} \qquad \blacksquare$$

We will not prove this proposition formally, but will rather develop an informal argument using (2.5-10) for the case N = 1. Since (2.5-10) is nonlinear in F, it is not a straightforward matter to perform a general analysis of this system of dynamic matrix equations. When N = 1, x_S^o and F_1 are scalars, $\bar{x}_S^{-1}$, F_{12}, and F_2 are 2 x 1, 1 x 2, and 2 x 2, respectively. As $|x_S^o(k)| \to \infty$, terms quadratic in x_S^o dominate, and (2.5-10) becomes:

$$F_1(k+1) = F_1(k) - \frac{F_1{}^2(x_S^o)^2}{F_1(x_S^o)^2} = F_1 - F_1 = 0 \tag{2.5-12a}$$

$$F_{12}(k+1) = F_{12}(k) - \frac{F_1(x_S^o)^2F_{12}}{F_1(x_S^o)^2} = F_{12} - F_{12} = 0 \tag{2.5-12b}$$

$$F_2(k+1) = F_2(k) - \frac{(x_S^o)^2F_{12}^TF_{12}}{F_1(x_S^o)^2} = F_2 - \frac{F_{12}^TF_{12}}{F_1}. \tag{2.5-12c}$$

Proposition 2.5.1, in conjunction with (2.5-11), leads to the observation that when the MRAS exhibits instability, S-subsystem pole adaptation is effectively halted. As $\|x_S^o(k)\|$ becomes unbounded, one observes in simulations of (2.5-7) that F_1 and F_{12} converge rapidly to 0, preventing $a(k)$ from possibly returning to a region corresponding to stable poles. We should emphasize, however, that this property of the dynamic adaptation gain matrix is not the cause for MRAS instability. We have suggested that this instability is due to the loss of positive-realness in (2.5-1). What is significant, however, is that the MRAS having entered into an unstable mode of operation, coupled with the property of $F(k)$ under such conditions, creates a situation which appears to reinforce the instability. One should examine whether using a fixed gain $F = F^T > 0$ might improve the stability properties of the MRAS when using (2.5-4).

We have concluded that potential instability of the Landau MRAS is due to the difficulty of selecting the vector c so that (2.5-1) is guaranteed positive-real over the entire time of MRAS operation. The vector c was introduced into the MRAS structure in the definition

$$v(k) = c^T e(k) + \varepsilon(k) \tag{2.5-13}$$

In concluding this section we will examine this equation, first considering why it is necessary to include a term involving the output-error. Then we will consider reformulating the MRAS structure in such a way as to eliminate the term involving the unknown vector c. By doing this, we eliminate the cause for potential instability, and obtain a "robustly" hyperstable MRAS, one whose stability does not depend on the value of a design-variable.

The output error term $\varepsilon(k) = y_M(k) - y_S(k)$ in (2.5-13) may be expressed:

$$\varepsilon(k) = a_M^T e(k) + \bar{x}_S^T(k)\ \delta\bar{p}(k+1) = a_M^T e(k) + w(k) \tag{2.5-14}$$

and so the Popov linear subsystem in Figure 2.1-3 appears as

$$e(k+1) = F_M e(k) + G_2\ w(k) \tag{2.5-15}$$

$$v(k) = (a_M+c)^T e(k) + w(k) \tag{2.5-16}$$

Thus we see that the term $\varepsilon(k)$ in (2.5-13) is responsible for introducing the feedforward element in the Popov linear subsystem. Excluding $\varepsilon(k)$ in (2.5-13) would yield (2.5-16) in the form:

$$v(k) = (a_M+c)^T e(k) \tag{2.5-17}$$

We make use of the discrete positive-real lemma [16] in the following proposition

Proposition 2.5.2

A SISO positive-real linear system of the form

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = C^T x(k) + Du(k)$$

necessarily contains a non-zero feedforward element $D \neq 0$.

Proof: According to the discrete positive-real lemma, necessary and sufficient conditions for a linear system to be positive-real are that there exist $P = P^T > 0$, L, and W such that

1) $A^T PA - P = -LL^T$

2) $A^T PB = \underline{C} - LW$

3) $W^T W = D + D^T - B^T PB$

If $D = 0$, condition (3) cannot be satisfied, since $W^T W \geq 0$, and $-B^T PB < 0$ since $P > 0$. ■

This proposition implies that $v(k)$ necessarily contains a linear term in $\varepsilon(k)$, given the nonlinear-subsystem output, $z(k)$, in order for the linear subsystem to contain a feedforward element. This is in contrast to the continuous-time hyperstable parallel MRAS structure, where it is possible for a stable MRAS to have an adaptation equation of the form

$$\dot{p}(t) = F\,\bar{x}_S(t)[C^T e(t)], \tag{2.5-18}$$

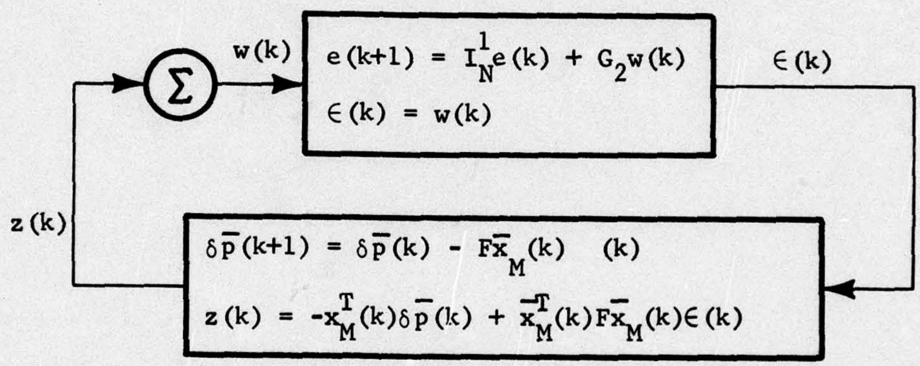
i.e., simply using the state error as an input to the adaptation subsystem. In the discrete-time case, the term $\varepsilon(k)$ must be included as well. One

might interpret this difference by saying that the parameter change $\Delta\overline{p}(k) = F\,\overline{x}_S v(k)$ must take into account the effect of parameter change on the system at the time stage k by including $\varepsilon(k)$ as part of v(k). In the limit, as the sample time $T \to 0$, $\varepsilon(k) \to \varepsilon(k-1)$, and thus $\varepsilon(k)$ conveys no information in addition to the state-error vector, hence is not necessary in the continuous-time problem. (Note that $\varepsilon(k-1)$ is an element of the state-error vector, as defined in 2.1-5.)

Discrete-time MRAS identification algorithms have been presented [19, 29] which are asymptotically stable in the state-error and which contain no design parameters depending on the unknown plant parameters $\overline{p}_M$. Ionescu and Monopoli achieve a stable parallel MRAS design by introducing an augmented error signal generated by an augmentation filter. This approach seems to be related to the augmented state-variable filter approach for the continuous-time MRAS problem [40]. Kudva and Narendra propose a stable series-parallel MRAS which is designed from the Liapunov viewpoint, without requiring knowledge of plant parameters, and requiring only the state-error $e(k) = x_M(k) - x_S(k)$ as an input to the adaptation subsystem. The Landau MRAS structure on which we have focused thus far can also be reformulated into a series-parallel context, resulting in a hyperstable system with an adaptation equation

$$\overline{p}(k+1) = \overline{p}(k) + F(k)\,\overline{x}_M(k)\,\varepsilon(k) \tag{2.5-19}$$

The equivalent Popov structure for this series-parallel MRAS is shown in Figure 2.5-1. The linear subsystem is always positive-real without having to



I_N^1 and G_2 are defined in Section 2.1

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Figure 2.5-1 Equivalent Popov Structure for Landau Series-Parallel MRAS

introduce additional design parameters, as was necessary in the parallel MRAS case, eliminating the problem of potential instability.

This stability design advantage, however is offset by a disadvantage relating to measurement noise. The adaptation process (2.5-19) differs from the equivalent equation for the parallel MRAS in that it uses plant output measurements rather than identification model output measurements ($\bar{x}_M$ vs. $\bar{x}_S$). In a practical situation, the former is assumed to be relatively more noise-contaminated. The effect of this noise is to produce a biased parameter estimate in the series-parallel case, in contrast to an unbiased parameter estimate in the parallel case. In the identification of a real heat exchanger plant, it was demonstrated [15] that this is indeed what occurs when comparing the two MRAS structures.

Thus we conclude that the problem of potential instability of the Landau parallel MRAS identification algorithm may be confronted from one of several directions. One may retain the given parallel structure, and endeavor to select the design vector c in a way which retains long-term MRAS hyperstability. It may be necessary to use a pre-identifier, such as a least-squares method or a series-parallel MRAS, to initialize the parallel MRAS. Another direction would be to consider using only the series-parallel MRAS structure. The cost in taking this approach, however, is accepting noise-biased parameter estimates.

2.6 Discontinuous Adaptation in MRAS

Up to this point we have considered continuous parameter adaptation, that is, adaptation of the type (2.1-14) which is a continuous function of the variables $\overline{x}_S(k)$ and $v(k)$. In the context of the MRAS control problem, our goal is not generally to achieve convergence of $\delta\overline{p}(k) = \overline{p}_M - \overline{p}(k)$ to the origin in parameter-error space, where $\overline{p}_M$ corresponds to reference model parameters, $\overline{p}(k)$ to the controlled plant parameters, but rather to provide only state-error convergence to the origin. Thus, continuous adaptation (i.e., the capability for $\overline{p}(k)$ to attain the value $\overline{p}_M$) may not be necessary. It may be desirable to consider simpler adaptation schemes which require, for example, less extensive multiplication hardware (or multiplication processing in a digital computer) than is necessary for the continuous parameter adaptation algorithm (2.1-14).

In this section we will consider a discontinuous approach to stable single-input single-output MRAS control systems design. This viewpoint has been studied primarily by researchers in the USSR [53, 65, 66, 26, 27, 28], but some research activity in this direction has also occurred in the U. S. [45, 38]. In this previous work, Liapunov-techniques are used both to obtain and to analyze stability conditions for discontinuous adaptation algorithms for MRAS control. A recent comprehensive treatment and analysis is [8]. One feature of the discontinuous adaptation MRAS to be studied here is that the plant parameter vector $\overline{p}(k)$ may take on one of only a finite set of values, the order of that set for the ELMA (see Section 2.2) being $2^{(2N+1)}$, where $\overline{p} \in R^{2N+1}$.

First we present the discontinuous parameter adaptation MRAS design for continuous-time systems, proving stability if a set of assumptions are satisfied. This problem is then transformed to the signal synthesis form, where a discontinuous control signal, rather than parameter adaptation, is used to achieve the asymptotic stability of the state-error. Two classes of signal synthesis are considered, which we shall call "direct" and "augmented." In both the parameter adaptation and signal synthesis forms we adopt notation for the adaptation process presented in [60].

Next we introduce a measurable disturbance-rejection problem developed in [60]. The control design is of the variable structure or discontinuous type. We find that the solution for this problem is identical in form to the discontinuous signal synthesis MRAS design, and show that this MRAS may be designed without resorting to the Liapunov approach. Sliding mode properties of this MRAS are discussed.

Finally, the above developments are applied to the discrete-time Landau MRAS or ELMA. It is proved through hyperstability considerations that discontinuous parameter adaptation provides stability if a set of assumptions are satisfied. The concept of sliding on the discontinuous surface for the discrete-time case must not be viewed literally, as was done for the continuous-time MRAS.

Discontinuous Parameter Adaptation

Consider the continuous time single-input single-output reference model and controlled plant, respectively:

$$H_M(S) = \frac{\sum_{i=0}^{m} b_{M_{n-m+i}} S^{m-i}}{S^n - \sum_{i=1}^{n} a_{M_i} S^{n-1}} \tag{2.6-1}$$

$$H_S(S) = \frac{\sum_{i=0}^{m} b_{S_{n-m+i}} S^{m-i}}{S^n - \sum_{i=1}^{n} a_{S_i} S^{n-i}} \tag{2.6-2}$$

where we assume $m \leq n$.

We may represent the two systems in a non-minimal state-variable form as:

$$\begin{aligned} \dot{x}_M &= \hat{A}_M x_M + \hat{B}_M v^{(m)} \\ y_M &= [1\ 0\ \dots\ 0]\ x_M \end{aligned} \tag{2.6-3}$$

$$\begin{aligned} \dot{x}_S &= \hat{A}_S(t) x_S + \hat{B}_S(t)\ v^{(m)} \\ y_S &= [1\ 0\ \dots\ 0]\ x_S \end{aligned} \tag{2.6-4}$$

where $x^T = [x^{o^T} \vdots x^1] = [y\ \dot{y}\ \dots\ y^{(n-1)} \vdots v\ \dot{v}\ \dots\ v^{(m-1)}]$ (2.6-5)

$$\hat{A} = \left[\begin{array}{c|c} A & \hat{F}(1) \\ \hline 0 & I_m' \end{array}\right] = \left[\begin{array}{cccc|ccccc} 0 & 1 & \dots & 0 & 0 & \dots & & & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \dots & & 1 & 0 & \dots & & & 0 \\ & a^T & & & & & b^T & & \\ \hline & & & & 0 & 1 & 0 & \dots & 0 \\ & 0 & & & \vdots & & 1 & & \\ & & & & 0 & \dots & & 0 & 1 \\ & & & & 0 & \dots & & & 0 \end{array}\right] \tag{2.6-6}$$

$$\hat{B}^T = [B_1^T \vdots B_2^T] = [0 \; \ldots \; 0 \; b_{n-m} \vdots 0 \; \ldots \; 01] \tag{2.6-7}$$

$$\bar{p}^T = [a^T \; b^T] = [a_n \; \ldots \; a_1 \vdots b_n \; \ldots \; b_{n-m+1} \vdots b_{n-m}] \tag{2.6-8}$$

with the appropriate subscripts (M or S) attached.

Note that this state-variable structure is the continuous-time analog to the I-O Delay structure in Section 2.2. By defining an error vector:

$$e = x_M^o - x_S^o \tag{2.6-9}$$

the dynamic error system becomes:

$$\dot{e} = A_M e + B_2(\delta\bar{p}^T(t)\bar{x}_S(t)) \tag{2.6-10}$$

where $\delta\bar{p}(t) = \bar{p}_M - \bar{p}_S(t)$ (2.6-11)

$$\bar{x}_S^T(t) = [x_S^T \vdots v^{(m)}] \tag{2.6-12}$$

Note that x_M^o and x_S^o are <u>not</u> the states of the reference model and plant.

<u>Theorem 2.6.1</u> There exists a vector $c \in R^n$ and matrix $\Lambda = \text{diag}\,[\alpha_i]$ such that the parameter adaptation algorithm:

$$\bar{p}_S(t) = \bar{p}_S(0) + \Lambda \,\text{sgn}\,(\bar{x}_S)\,\text{sgn}\,(c^T e)$$

provides an asymptotically stable MRAS control system; that is, $\lim_{t\to\infty} e(t) = 0$. The notation sgn $(\bar{x}_S)$ means

$$[\text{sgn}\,(\bar{x}_{S1}) \; \ldots \; \text{sgn}\,(\bar{x}_{S_{n+m+1}})]^T.$$

<u>Proof</u> Consider the scalar function $V = e^T P e$, $P = P^T > 0$. Then

$$\dot{V} = e^T(A_M^T P + PA_M)e + 2(c^T e)(\overline{x}_S^T \delta\overline{p}(t))$$

$$= - e^T Q\, e + 2(c^T e)(\overline{x}_S \delta\overline{p})$$

where $Q > 0$, and c is the last column of P. Since A_M is assumed stable, such a positive-definite pair (P,Q) exists. Thus, if $(c^T e)\overline{x}_S{}^T \delta\overline{p})$ can be shows to be negative, V is a Liapunov function, and $e = 0$ is asymptotically stable. Substituting the algorithm for $\overline{p}_S(t)$, we obtain

$$(c^T e)(\overline{x}_S^T \delta\overline{p}) = |c^T e| \sum_{i=1}^{n+m+1} |\overline{x}_{S_i}| (\delta p_i(0)\ \mathrm{sgn}(\overline{x}_{S_i})\ \mathrm{sgn}\ (c^T e) - \alpha_i)$$

If $\alpha_i \geq |\delta\overline{p}_i(0)|$, then $\dot{V} \leq 0$, with equality possible only at $e = 0$ ■

Thus, such a discontinuous adaptation algorithm guarantees asymptotic model-following if we can bound each element of the initial parameter error $\delta\overline{p}(0)$. This condition implies that one must know bounds on the unknown plant parameters, $\overline{p}_{i_{min}} \leq \overline{p}_i(0) \leq \overline{p}_{i_{max}}$. This design may exhibit a stable sliding motion on the surface $\sigma = c^T e = 0$. Sliding is a property whereby the system trajectory e(t) reaches the discontinuity surface $\sigma = 0$ and then moves along that surface while the control variable sgn(σ) oscillates at a high frequency. Necessary and sufficient conditions for sliding to exist are [60]:

$$\mathrm{sgn}\ (\dot{\sigma}) = -\ \mathrm{sgn}\ (\sigma) \tag{2.6-13}$$

or equivalently,

$$\sigma\, \dot{\sigma} < 0 \tag{2.6-14}$$

In this case

$$\dot{\sigma} = c^T\dot{e} = c^T A_M e + c_n \bar{x}_S^T \delta p(0) - c_n \sum_{i=1}^{n+m+1} \alpha_i |\bar{x}_{S_i}| \operatorname{sgn}(\sigma) \tag{2.6-15}$$

Thus (2.6-14) will be satisfied, and sliding motion will occur when:

a) $\sigma = 0$

b) $\displaystyle\sum_{i=1}^{n+m+1} \alpha_i |\bar{x}_{S_i}| \geq \frac{c^T A_M e}{c_n} + \bar{x}_S^T \delta p(0)$

in some neighborhood of $\sigma = 0$.

If sliding motion occurs, the trajectory for the error vector is determined by the differential equation

$$\dot{\tilde{e}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & 1 \\ -c_1/c_n & \cdots & & & c_{n-1}/c_n \end{bmatrix} \tilde{e} \tag{2.6-16}$$

where $\quad \tilde{e}^T = [e_1 \ \cdots \ e_{n-1}]$ (2.6-17)

This result is easily obtained when one recognizes that during sliding, $\sigma = c^T e = 0$, or $e_n = -\frac{1}{c_n}\sum_{i=1}^{n-1} c_i e_i$. It may be shown [7, 8] that for c chosen as the last row of P, (2.6-16) is asymptotically stable, i.e., stable sliding will occur.

Using the notation in [60], the discontinuous parameter adaptation algorithm may be expressed:

$$\bar{p}(t) = \bar{p}(0) + \Psi \tag{2.6-18}$$

where $\psi^T = [\psi_1 \; \dots \; \psi_{n+m+1}]$ (2.6-19

$$\psi_i = \begin{cases} \alpha_i & \text{if } \bar{x}_{S_i} \sigma > 0 \\ -\alpha_i & \text{if } \bar{x}_{S_i} \sigma < 0 \end{cases} \quad (2.6\text{-}20)$$

$$\sigma = c^T e \quad (2.6\text{-}21)$$

Throughout the remainder of this section we will use the notation (2.6-20) wherever possible.

Discontinuous Signal Synthesis

As an alternative to the parameter adaptation approach, the MRAS concept may be implemented using a signal synthesis approach [34]. Either of two variations may be used, which we will call augmented and direct signal synthesis. Figure 2.6-1 depicts the two variations. We will now derive the signal synthesis MRAS controls $\Delta u(t)$ and $u(t)$ which would be equivalent to the discontinuous parameter adaptation algorithm (2.6-18-2.6-21).

Redefine the plant (2.6-4) to be

$$\begin{aligned} \dot{x}_S &= \hat{A}_S x_S + \hat{B}\, u^{(m)} \\ y_S &= [1 \; 0 \; \dots \; 0]\, x_S \end{aligned} \quad (2.6\text{-}22)$$

where now

$$x_S^T = [x_S^{o\,T} \vdots x_S^{1\,T}] = [y_S \; \dots \; y_S^{(n-1)} \vdots u \; \dot{u} \; \dots \; u^{(m-1)}] \quad (2.6\text{-}23)$$

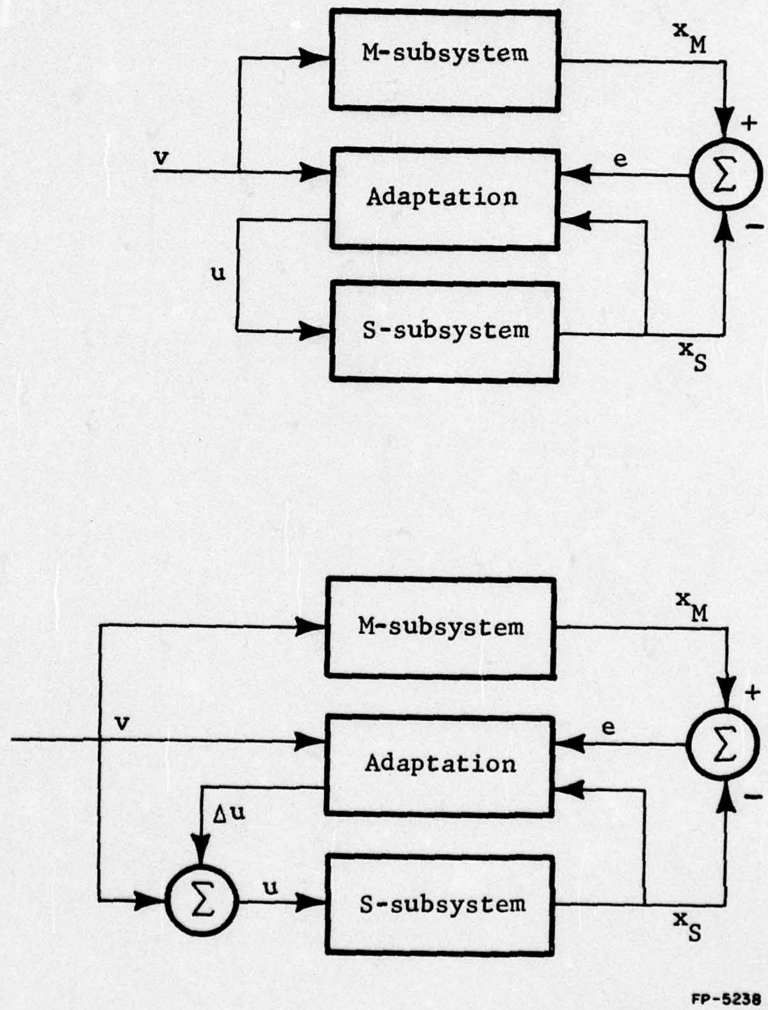

Figure 2.6-1 Two Variations of the Signal Synthesis MRAS

The system matrices $\hat{A}_S$ and $\hat{B}_S$ are assumed now to be constant during the operation of the MRAS. For the direct signal synthesis case, adaptation will be achieved directly through u(t). In the augmented case , we define

$$u(t) = v(t) + \Delta u(t) \tag{2.6-24}$$

with adaptation being achieved through $\Delta u(t)$. The system error equation for the direct and augmented control cases are, respectively:

$$\dot{e} = A_M e + B_2\{\delta \bar{a}^T x_S^o + b_M^T x_M^1 - b_S^T x_S^1 - b_{S_{n-m}} u^{(m)}\} \tag{2.6-25a}$$

$$\dot{e} = A_M e + B_2\{\delta \bar{a}^T x_S^o + b^T \bar{x}_M^1 - b_S^T \Delta\bar{u} - b_{S_{n-m}} \Delta u^{(m)}\} \tag{2.6-25b}$$

where $\Delta\bar{u} = x_S^1 - x_M^1$ (2.6-26)

Equating (2.6-25) with (2.6-10), we obtain:

$$u^{(m)} = \mathrm{sgn}(b_{n-m})\{\psi^{o^T} x_S^o + \psi_M^{1^T} \bar{x}_M^1 + \psi_S^{1^T} x_S^1\} \tag{2.6-27a}$$

where

$$\psi_i^o = \begin{cases} \alpha_i^o & \text{if } x_{S_i}^o \sigma > 0 \\ -\alpha_i^o & \text{if } x_{S_i}^o \sigma < 0 \end{cases} \qquad i = 1, \ldots, n$$

$$\psi_{M_i}^1 = \begin{cases} \alpha_{M_i}^1 & \text{if } \bar{x}_{M_i}^1 \sigma > 0 \\ \alpha_{M_i}^1 & \text{if } \bar{x}_{M_i}^1 \sigma < 0 \end{cases} \qquad i = 1, \ldots, m + 1$$

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$$\psi_S^1 = \begin{cases} \alpha_{S_i}^1 & \text{if } x_{S_i}^1 \sigma > 0 \\ -\alpha_{S_i}^1 & \text{if } x_{S_i}^1 \sigma < 0 \end{cases} \qquad i = 1, \ldots, m$$

$$\alpha_i^o \geq \left| \frac{\delta a_{n-(i-1)}}{b_{n-m}} \right|$$

$$\alpha_{M_i}^1 \geq \left| \frac{b_{M_{n-(i-1)}}}{b_{n-m}} \right|$$

$$\alpha_{S_i}^1 \geq \left| \frac{b_{n-(i-1)}}{b_{n-m}} \right|$$

$$\Delta u^{(m)} = \text{sgn}(b_{n-m})\{\psi^{o^T} x_X^o + \psi_M^{1^T} \bar{x}_M^1 + \psi_S^{1^T} \Delta\bar{u}\}$$

where
$$\psi_i^o = \begin{cases} \alpha_i^o & \text{if } x_{S_i}^o \sigma > 0 \\ -\alpha_i^o & \text{if } x_{S_i}^o \sigma < 0 \end{cases} \qquad i = 1, \ldots, n$$

$$\psi_{M_i}^1 = \begin{cases} \alpha_{M_i}^1 & \text{if } \bar{x}_{M_i}^1 \sigma > 0 \\ -\alpha_{M_i}^1 & \text{if } \bar{x}_{M_i}^1 \sigma < 0 \end{cases} \qquad i = 1, \ldots, m + 1$$

$$\psi_{S_i}^1 = \begin{cases} \alpha_{S_i}^1 & \text{if } \Delta\bar{u}_i \sigma > 0 \\ -\alpha_{S_i}^1 & \text{if } \Delta\bar{u}_i \sigma < 0 \end{cases} \qquad i = 1, \ldots, m$$

and $$\alpha_i^o \geq \left| \frac{\delta a_{n-(i-1)}}{b_{n-m}} \right|$$

$$\alpha_{M_i}^1 \geq \left| \frac{\delta b_{n-(i-1)}}{b_{n-m}} \right|$$

$$\alpha_{S_i}^1 \geq \left| \frac{b_{n-(i-1)}}{b_{n-m}} \right| \qquad (2.6\text{-}27b)$$

The terms $\psi_S^{1^T} x_S^1$ and $\psi_S^{1^T} \Delta\bar{u}$ in (2.6-27) are substituted for the naturally occurring terms $\frac{b_S^T}{b_{n-m}} x_S^1$ and $\frac{b_S^T}{b_{n-m}} \Delta\bar{u}$, respectively, since b_S is part of the unknown plant parameter vector (2.6-8). As long as $\alpha_{S_i}^1$, $i = 1, \ldots, m$, satisfy the bounds indicated above, it may be shown through Liapunov stability analysis that the stability property of the MRAS is retained.

The stable discontinuous signal synthesis MRAS design introduces constraints on the plant zeroes and the discontinuty surface coefficient c_n which do not appear either in the discontinuous parameter adaptation design or in any of the continuous adaptation designs. These constraints are introduced by the terms $b_S^T x_S^1$ and $b_S^T \Delta\bar{u}$ in (2.6-25), and become significant to system stability only when the MRAS exhibits a sliding motion. We analyze the augmented signal synthesis case here; the direct case exhibits similar properties. This aspect of MRAS design was first considered in [64] for the case when $c_n = 1$.

During sliding, the condition $\sigma = \dot{\sigma} = 0$ and (2.6-25b) provide us with an equivalent augmented control [60], denoted $\Delta u_{eq}^{(m)}$:

$$\Delta u_{eq}^{(m)} = \frac{1}{b_{n-m}} \left\{ \frac{c^T A_M e}{c_n} + \delta a^T x_S^o + \delta b^T x_M^1 - b_S^T \Delta\bar{u} \right\} \qquad (2.6\text{-}28)$$

This control corresponds to the effective continuous value of $\Delta u^{(m)}$ necessary to maintain motion of e(t) along $\sigma = 0$. The motion of $\overline{\Delta u}$, defined by (2.6-26), during sliding is determined by:

$$\dot{\overline{\Delta u}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & & \dots & 1 \\ & -c_n\left(\frac{b_S^T}{b_{n-m}}\right) & & \end{bmatrix} \overline{\Delta u} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{c_n}{b_{n-m}} \end{bmatrix} \left\{\frac{c^T A_M e}{c_n} + \delta a^T x_S^o + \delta b^T x_M^1\right\} \tag{2.6-29}$$

Considering $\overline{\Delta u}_1$ as the output of this subsystem, the transfer function from $\Delta u_{eq}^{(m)}$ to $\overline{\Delta u}_1$ is:

$$H(S) = \frac{\frac{1}{b_{n-m}}}{\left(\frac{1}{c_n}\right)S^m + \frac{b_{n-m+1}}{b_{n-m}} S^{m-1} + \dots + \frac{b_n}{b_{n-m}}} \tag{2.6-30}$$

i.e., the subsystem poles of (2.6-30) are shifted from the plant zeroes of (2.6-2) by an amount dependent upon the value of (c_n-1). Thus we are required to place the following restrictions on the plant zeroes occurring in (2.6-2) and on c_n in order to guarantee asymptotic stability of the MRAS during sliding:

a) The plant numerator coefficient vector b_S must be such that a positive scalar c_n exists so the poles of (2.6-30) are asymptotically stable. Necessary conditions for the existence of c_n are [61]:

i) No elements of b_S may be zero,

ii) All elements of $b_S^T = [b_n \ldots b_{n-m+1}]$ must have same sign.

b) given that b_S satisfies (a), c_n must be chosen to place the poles of (2.6-30) in the strict left-hand s-plane.

To illustrate these constraints, we consider three examples for the case m = 3, examining the effect c_n may have on various systems of the form (2.6-30).

Example 2.6.7

a) $$H(S) = \frac{1}{(\frac{1}{c_n})S^3 + 3S^2 + 4S + 2}$$

This case corresponds (for $c_n = 1$) to plant zeroes in the left-half s-plane at s = -1, s = - 1 ± j. Figure 2.6-2(a) illustrates that for $c_n > 0.167$, the poles of H9s) are asymptotically stable, but when $c_n \leq 0.167$, the complex poles move into the right-hand s-plane. In the limit as $c_n \to \infty$, a 2nd order system is obtained with poles at s = - .67 ± j.47. Thus constraint (a) is satisfied, and it is possible to design a stable discontinuous MRAS by choosing $c_n > 0.167$.

b) $$H(S) = \frac{1}{(\frac{1}{c_n})S^3 + S^2 + 1.25S + 6.5}$$

This case corresponds to plant zeroes at S = -2, S = + .5 ± $j\sqrt{3}$, yielding an unstable subsystem H(S) when $c_n = 1$. However, if $c_n > 5$, the unstable poles shift into the asymptotically stable region (Figure 2.6-2(b)). In the limit as $c_n \to \infty$, a 2nd order system is obtained with poles at S = - .625 ± j 2.48.

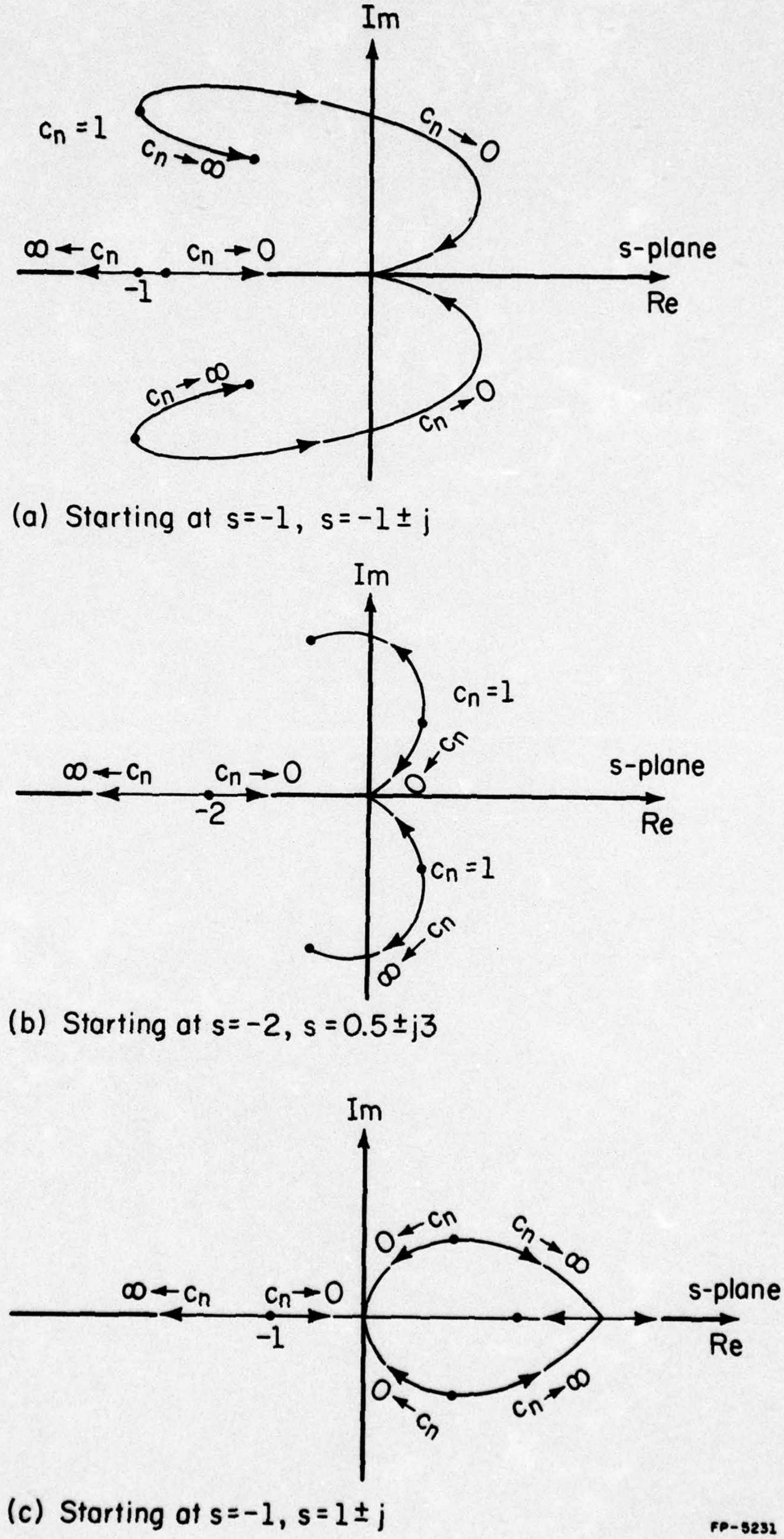


Figure 2.6-2 c_n-locus of Roots for 3^{rd} Order Systems

Again, constraint (a) is satisfied, despite the fact that the original plant is non-minimum-phase.

c) $H(S) = \dfrac{1}{(\frac{1}{c_n})S^3 - S^2 + 2}$

This case corresponds to plant zeroes at $s = -1$, $s = +1 \pm j$. Since it violates both necessary conditions in constraint (a), the unstable poles remain unstable for all values of c_n, as seen in Figure 2.6-2(c). In the limit as $c_n \to \infty$, an unstable 2nd order system is obtained with poles at $S = \pm\sqrt{2}$. ■

Relation between Disturbance Rejection in VSS and the Discontinuous MRAS

Variable structure system (VSS) design theory [60] has been developed to solve the problem of rejecting both measurable and unmeasurable input disturbances. Consider the stable single-input phase-canonic plant:

$$\dot{x} = Ax + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (f-w) \qquad (2.6\text{-}31)$$

where w is the control and f is a measurable disturbance. A variable structure control design which rejects the disturbance f, thus allowing x to approach the origin asymptotically, is:

$$w = \psi f$$

where
$$\psi = \begin{cases} \alpha & \text{if } f\sigma > 0 \\ \beta & \text{if } f\sigma < 0 \end{cases}$$

$$\sigma = c^T x$$

$$\alpha > 1$$

$$\beta < 1$$

c chosen to provide stable sliding motion.

$$c_n = 1 \qquad (2.6\text{-}32)$$

When sliding occurs, the equivalent control, w_{eq}, is

$$w_{eq} = \psi_{eq} f = (1 + \frac{c^T A x}{f}) f \qquad (2.6\text{-}33)$$

and the motion of x along $\sigma = 0$ is described by:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \dots & 0 \\ \vdots & & \\ 0 & \dots & 1 \\ -c_1 & \dots & -c_{n-1} \end{bmatrix} \bar{x} \qquad (2.6\text{-}34)$$

where $\bar{x}^T - [x_1 \dots x_{n-1}]$ (2.6-35)

since $x_n = - \sum_{i=1}^{n-1} c_i x_i$. As x slides toward the origin along $\sigma = 0$, we see from (2.6-33) that w_{eq} approaches f; thus this design exactly counteracts the disturbance at the equilibrium point $x = 0$.

The existence condition for sliding is $\sigma \dot{\sigma} < 0$.

$$\dot{\sigma} = c^T Ax + (1-\psi)f \tag{2.6-36}$$

$$\sigma\dot{\sigma} = \begin{cases} [c^T Axx^T c] + (1-\alpha)\sigma f & \text{if } \sigma f > 0 \\ [c^T Axx^T c] + (1-\beta)\sigma f & \text{if } \sigma f < 0 \end{cases} \tag{2.6-37}$$

and so the existence condition for sliding is satisfied if:

$$\alpha > 1 + \frac{c^T Ax}{f} \tag{2.6-38a}$$

$$\beta < 1 + \frac{c^T Ax}{f} \tag{2.6-38b}$$

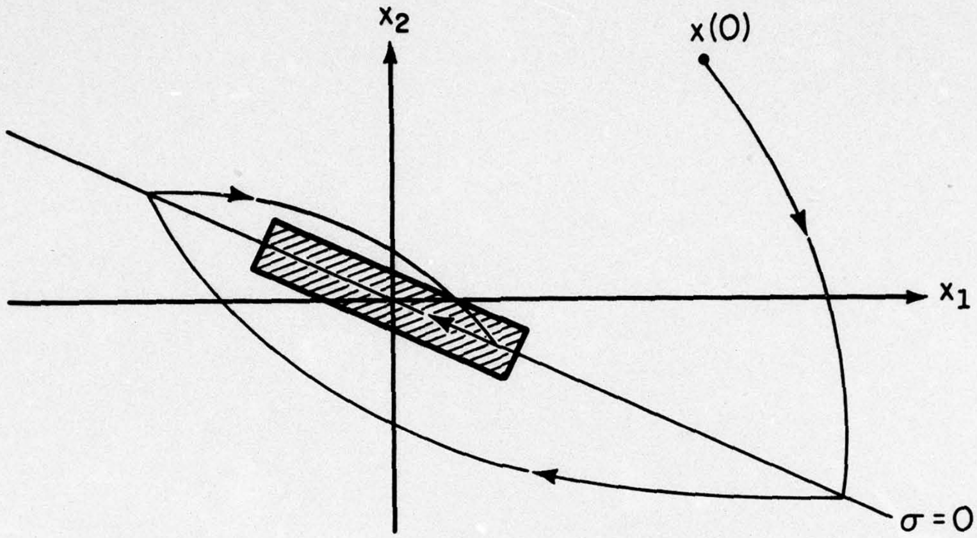
For a particular choice of $\alpha > 1$, $\beta < 1$, the existence conditions for sliding will be satisfied when the plant state and disturbance satisfy:

$$-(1-\beta) < \frac{c^T Ax}{f} \quad (\alpha-1) \tag{2.6-39}$$

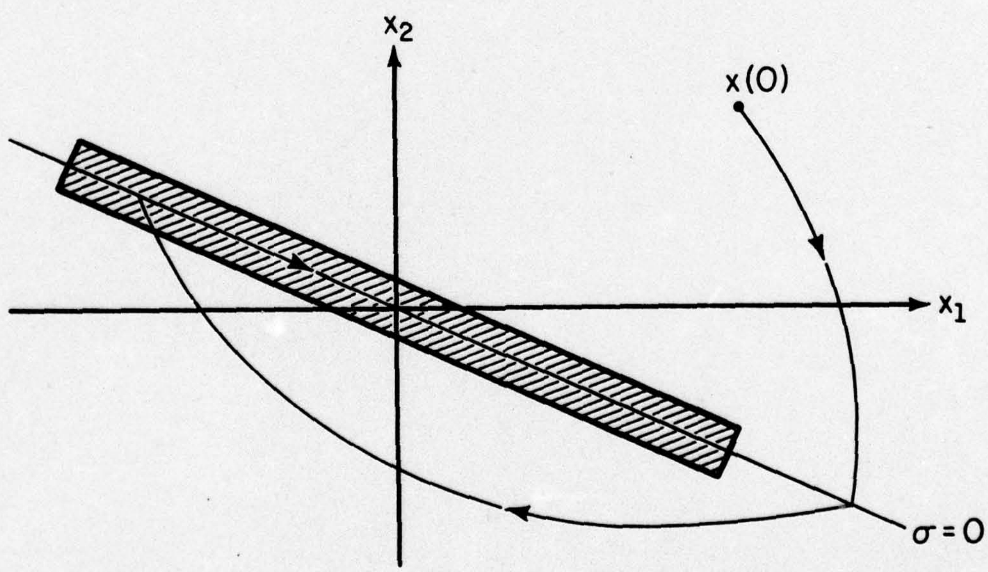
Thus sliding will occur when $\|x\|$ is "small enough" or if $|f|$ is "large enough"; i.e., the larger the disturbance magnitude, the greater the region along $\sigma = 0$ for which sliding will occur. Figure 6.2-3 illustrates this idea for two constant disturbances of different magnitudes, for given α and β.

An important feature of this design is that sliding always exists when $x = 0$; that is, the disturbance f will not force x away from the origin. At the origin the control exactly counteracts the disturbance and (2.6-39) is always satisfied for every $\alpha > 1$, $\beta < 1$.

If we examine the error equation (2.6-25) for the signal synthesis forms of the MRAS, we see that they have the same form as (2.6-31) by making the substitutions:



(a) Constant |f| "small"; yields small region where sliding exists



(b) Constant |f| "large"; yields large region near $\sigma = 0$ where sliding exists.

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Figure 2.6-3 Regions of Sliding for Disturbance-Rejection VSS

$$\left.\begin{array}{l} b_{n-m}u^{(m)} \\ b_{n-m}\Delta u^{(m)} \end{array}\right\} = w \tag{2.6-40}$$

and

$$\left.\begin{array}{l} \delta a^T x_S^o + \bar{b}_M^T \bar{x}_M^1 - b_S^T x_S^1 \\ \delta a^T x_S^o + \delta \bar{b}^T \bar{x}_M^1 - b_S^T \overline{\Delta u} \end{array}\right\} = f_1 + f_2 + f_3 = f \tag{2.6-41}$$

We will consider only the augmented signal synthesis case from this point. Following the design principle of (2.6-32), the discontinuous MRAS control $b_{n-m}\Delta u^{(m)}$ will reject the disturbance (2.6-41) when

$$b_{n-m}\Delta u^{(m)} = \psi^1 f_1 + \psi^2 f_2 + \psi^3 f_3$$

where

$$\psi^i = \begin{cases} \alpha^i & \text{if } f_i\sigma > 0 \\ \beta^i & \text{if } f_i\sigma < 0 \end{cases}$$

$$\alpha^i > 1$$

$$\beta^i < 1$$

$$\sigma = c^T e \tag{2.6-42}$$

Since we cannot measure directly the complete disturbance signals, f_1, f_2, and f_3, in the MRAS, the design for $b_{n-m}\Delta u^{(m)}$ must be modified slightly, while still retaining the desirable disturbance-rejection property of the VSS.

The following theorem suggests the solution to this problem for the MRAS.

Theorem 2.6.2 Given the stable plant (2.6-31), but with $f = \gamma\overline{f}$, $\overline{f}$ measurable and γ unknown but bounded by $\gamma_{min} \leq \gamma \leq \gamma_{max}$, then the control:

$$\overline{w} = \psi'\overline{f}$$

$$\psi' = \begin{cases} \alpha' & \text{if } \sigma\overline{f} > 0 \\ \beta' & \text{if } \sigma\overline{f} < 0 \end{cases}$$

$$\alpha' > \gamma_{max}$$

$$\beta' < \gamma_{min}$$

retains the disturbance-rejection property provided by (2.6-32).

Proof: The system may be expressed:

$$\dot{x} = Ax + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma \end{bmatrix} (\overline{f} - \frac{\overline{w}}{\gamma})$$

Assuming that the control

$$w = \frac{\overline{w}}{\gamma} = \psi\overline{f}$$

$$\psi = \begin{cases} \alpha > 1 & \text{if } \sigma\overline{f} > 0 \\ \beta < 1 & \text{if } \sigma\overline{f} < 0 \end{cases}$$

provides disturbance-rejection, then the control:

$$\overline{w} = \gamma\psi\overline{f} = \psi'\overline{f}$$

where

$$\psi' = \begin{cases} \alpha' > \gamma \text{ if } \sigma\overline{f} > 0 \\ \beta' < \gamma \text{ if } \sigma\overline{f} < 0 \end{cases}$$

also provides disturbance rejection. Since γ is unknown, if we choose:

$$\psi' = \begin{cases} \alpha' > \gamma_{max} \\ \beta' < \gamma_{min} \end{cases}$$

the conditions for disturbance rejection are still satisfied. ■

Applying this theorem to the design (2.6-42) for the augmented signal synthesis MRAS, we obtain a result which is identical to the design already obtained in (2.6-27b) from Liapunov methods. Thus the discontinuous signal synthesis control design for MRAS obtained from Liapunov stability analysis is equivalent to the solution of a VSS measurable disturbance rejection problem.

The VSS design theory may be extended in the discontinuous MRAS design problem to provide for global existence of sliding for all $e \in R^n$. This may be accomplished by adding another term to the control equation (2.6-27b) of the form [60] $\psi^T e$. The elements of ψ satisfy:

$$\psi_i = \begin{cases} \alpha_i & \text{if } \sigma e_i > 0 \\ \beta_i & \text{if } \sigma e_i < 0 \end{cases}$$

$$\alpha_i \geq \max\{c_{i-1} + a_{n-(i-1)} - c_i(c_{n-1} + a_1)\}$$

$$\beta_i \leq \min\{c_{i-1} + a_{n-(i-1)} - c_i(c_{n-1} + a_1)\} \tag{2.6-43}$$

The Discrete-Time MRAS with Discontinuous Adaptation

The continuous-time development for MRAS with discontinuous adaptation has a counterpart for discrete-time systems. The concept of sliding, however, takes on a different interpretation. Given a discontinuity surface in the MRAS state-error space, $\sigma(k) = c^T e(k)$, "sliding" motion of e(k), if it exists, does not occur precisely on $\sigma(k) = 0$, but rather the motion stays inside a Δ-neighborhood of $\sigma(k) = 0$. The size of the Δ-neighborhood will depend on the magnitude of the discontinuous gains.

A discontinuous parameter adaptation algorithm for the Landau MRAS or ELMA which is a counterpart to the continuous-time algorithm of Theorem 2.6.1 may be shown to be hyperstable, given that a set of inequalities is satisfied.

Theorem 2.6.3 Given the Landau MRAS of Figure 2.1-2, there exist a vector c and a matrix $\Lambda = \text{diag}[\alpha_i]$ such that the parameter adaptation algorithm:

$$\overline{p}(k+1) = \overline{p}(0) + \Lambda \text{ sgn } (\overline{x}_S) \text{ sgn } (v)$$

where

$$v = c^T e + \varepsilon$$

provides a hyperstable MRAS.

<u>Proof:</u> Transforming the MRAS into the Popov linear--nonlinear feedback structure, we obtain the system of Figure 2.6-4. The linear subsystem is identical to that for the continuous parameter adaptation design, and the vector $c = -a_M$ clearly makes the linear subsystem strictly positive-real. We must show that the nonlinear subsystem satisfies the Popov inequality:

$$\sum_{k=0}^{K} w(k)\ v(k) > -\gamma_o^2 \qquad *$$

$$\text{for all } K \geq 0,\ \gamma_o^2 > 0$$

The nonlinear subsystem output w(k) may be expressed:

$$w(k) = \sum_{i=1}^{2N+1} |\bar{x}_{S_i}| [\alpha_i \operatorname{sgn}(v) - \delta\bar{p}_i(0) \operatorname{sgn}(\bar{x}_{S_i})]$$

Thus, $w(k)\ v(k) = |v(k)| \sum_{i=1}^{2N+1} [\alpha_i - \delta\bar{p}_i(0) \operatorname{sgn}(\bar{x}_{S_i}) \operatorname{sgn}(v)]$

A sufficient condition for w(k) v(k) to be positive is:

$$\alpha_i \geq |\delta\bar{p}_i(0)|,\ i = 1, \ldots, 2N + 1 \qquad **$$

Therefore, the inequality * is satisfied whenever Λ satisfies **. ■

Thus in order to implement a discontinuous adaptation discrete-time MRAS, we must know bounds on elements of the parameter vector $\bar{p}_M$ (identification) or $\bar{p}_S(0)$ (control). This knowledge enables us to select elements for the gain matrix Λ which satisfy the condition ** in the theorem. Using the notation in [60], this adaptation algorithm may be expressed:

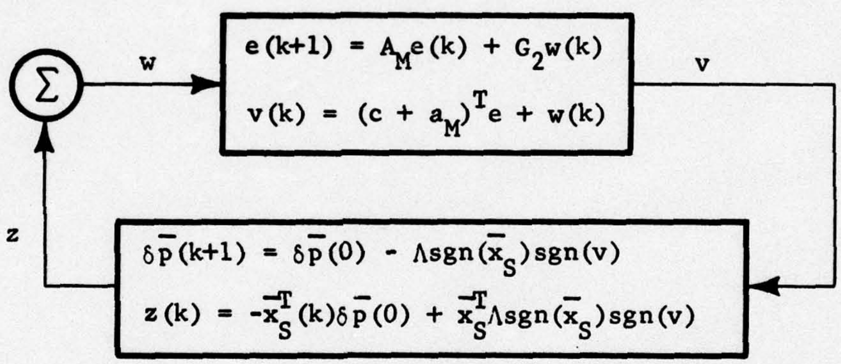

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Figure 2.6-4 Popov Structure for Discontinuous Parameter Adaptation Landau MRAS

$$\bar{p}(k+1) = \bar{p}(0) + \psi$$

$$\psi_i = \begin{cases} \alpha_i & \text{if } x_{S_i} v > 0 \\ -\alpha_i & \text{if } x_{S_i} v < 0 \end{cases}$$

$$\alpha_i \geq |\delta\bar{p}_i(0)| \tag{2.6-44}$$

Both direct and augmented signal synthesis versions of this MRAS algorithm may be derived in a way analogous to the continuous-time case. In the direct case, u(k) is the S-subsystem input; in the augmented case, $u(k) = z(k) + \Delta u(k)$ is the S-subsystem input, where z(k) is the M-subsystem input. The state-error equations are, respectively:

$$e(k+1) = A_M e(k) + G_2[\delta a^T x_S^1 + \bar{b}_M^T \bar{x}_M^2 - b_S^T x_S^2] - G_2\, b_{S_o}\, u(k) \tag{2.6-45a}$$

$$e(k+1) = A_M e(k) + G_2[\overline{\delta p}^T \bar{x}_S - b_S^T(x_S^2 - x_M^2)] - G_2\, b_{S_o}\, \Delta u(k) \tag{2.6-45b}$$

and the discontinuous control signals become:

$$u(k) = \text{sgn}(b_{S_o})\{\psi^{oT} x_S^o + \psi_M^{1^T} \bar{x}_M^1 + \psi_S^{1^T} x_S^1\}$$

$$\psi_i^o = \begin{cases} \alpha_i^o & \text{if } x_{S_i}^o \; v > 0 \\ -\alpha_i^o & \text{if } x_{S_i}^o \; v < 0 \\ \alpha_i^o \geq \left|\dfrac{\delta a_i}{b_{S_o}}\right|, & i = 1, \ldots, N \end{cases}$$

$$\psi_{M_i}^1 = \begin{cases} \alpha_{M_i}^1 & \text{if } \bar{x}_{M_i}^1 \; v > 0 \\ -\alpha_{M_i}^1 & \text{if } \bar{x}_{M_i}^1 \; v < 0 \end{cases}$$

$$\alpha^1_{M_i} \geq \left|\frac{b_{M_{N-(i-1)}}}{b_{S_o}}\right|, \; i = 1, \ldots N + 1$$

$$\psi^1_{S_i} = \begin{cases} \alpha^1_{S_i} & \text{if } x^1_{S_i} \; v > 0 \\ -\alpha^1_{S_i} & \text{if } x^1_{S_i} \; v < 0 \end{cases}$$

$$\alpha^1_{S_i} \geq \left|\frac{b_{S_{N-(i-1)}}}{b_{S_o}}\right|, \; i = 1, \ldots, N \qquad (2.6\text{-}46a)$$

$$\Delta u(k) = \mathrm{sgn}(b_{S_o})\{\psi^{o^T} \bar{x}_S + \psi^{1^T}(x^1_S - x^1_M)\}$$

$$\psi^o_i = \begin{cases} \alpha^o_i & \text{if } \bar{x}_{S_i} \, v > 0 \\ -\alpha^o_i & \text{if } \bar{x}_{S_i} \, v < 0 \end{cases}$$

$$\alpha^o_i \geq \left|\frac{\delta \bar{p}_i}{b_{S_o}}\right| \quad i = 1, \ldots, 2N + 1$$

$$\psi^1_i = \begin{cases} \alpha^1_i & \text{if } (x^1_{S_i} - x^1_{M_i})v > 0 \\ -\alpha^1_i & \text{if } (x^1_{S_i} - x^1_{M_i})v < 0 \end{cases}$$

$$\alpha^1_i \geq \left|\frac{b_{S_{N-(i-1)}}}{b_{S_o}}\right| \qquad (2.6\text{-}46b)$$

CHAPTER 3

THE MULTI-MODEL MRAS

This chapter introduces the concept of MRAS structures with multiple reference models, a class of MRAS which has apparently not appeared in the literature before. The idea for such a MRAS structure was suggested by the theory of dynamic differential games [20]. Differential games is often viewed as an extension or generalization of optimal control. A relationship between the single-model MRAS and an optimal regulator problem was developed in Chapter 1. This relationship, then serves as the motivation for jointly developing a multi-model MRAS and a dynamic game.

First we introduce the two-model MRAS (2M-MRAS), emphasizing the quasi-symmetry of this structure, and develop the motivation for the 2M-MRAS. The concept of a (2M-MRAS, dynamic game) pair is then developed. It is viewed by this author as a logical generalization of the related (MRAS, optimal regulator) pair developed in Chapter 1. Next we consider the question of stability for a discrete-time 2M-MRAS which uses the Landau hyperstable adaptation algorithm. A particular regulator-type (2M-MRAS, dynamic game) pair is then considered. Comparing the equilibrium solutions between the 2M-MRAS and the dynamic game for an assumed player strategy establishes a procedure for investigating possible equivalence between the two problems. By equivalence here we mean that the two equilibrium solutions are the same. Finally we analyze the effects of the quasi-symmetric elements on the equilibrium for the 2M-MRAS.

3.1 The 2M-MRAS

In this section we introduce the two-model MRAS (2M-MRAS), examine

its structure, and discuss the motivation for our interest in this class of MRAS.

Figure 3.1-1 depicts the basic 2M-MRAS parallel structure. It is similar to the one-model MRAS in Figure 2.1-1, with the addition of a second reference model (M_2) and adaptation algorithm (A_2). The S-subsystem parameter vector $\overline{p}(k)$ is influenced by the outputs of both A_1 and A_2. A quasi-symmetry is apparent in Figure 3.1-1 in the sense that the 2M-MRAS is symmetric at the level of detail depicted in Figure 3.1-1, but is not necessarily totally symmetric; $M_1 \neq M_2$, $A_1 \neq A_2$, and $u_1 \neq u_2$ in general. These quasi-symmetric elements may be used to classify the 2M-MRAS into subclasses. For example, two subclasses may be obtained depending on whether $u_1 = u_2$ or $u_1 \neq u_2$. Similarly classifications may be obtained using M_i and A_i, $i = 1, 2$. First we define two partial ordering relations [5], which may then be used to separate the 2M-MRAS into various subclasses.

<u>Definition 3.1.1</u> If $u_1 = u_2 = 0$, the subsystem states $x_{M_1}(0) = x_{M_2}(0)$, and the respective state trajectories satisfy the inequality:

$$\sum_{k=0}^{K} x_{M_2}^T(k)x_{M_2}(k) \geq \sum_{k=0}^{K} x_{M_1}^T(k)x_{M_1}(k) \text{ for all } K \geq 0$$

then we say that (M_1, M_2) satisfy the partial ordering $M_1 \stackrel{\geq}{\cdot} M_2$, and that M_1 is "globally uniformly faster" than M_2.

<u>Definition 3.1.2</u> For $u(k) \in U$, and for every 1M-MRAS compatible with A_α and A_β, if the 1M-MRAS with A_α and A_β, respectively, satisfies:

$$\{ \sum_{k=0}^{K} e^T(k)e(k) \mid A_\alpha\} \leq \{ \sum_{k=0}^{K} e^T(k)e(k) \mid A_\beta\} \text{ for all } K \geq 0,$$

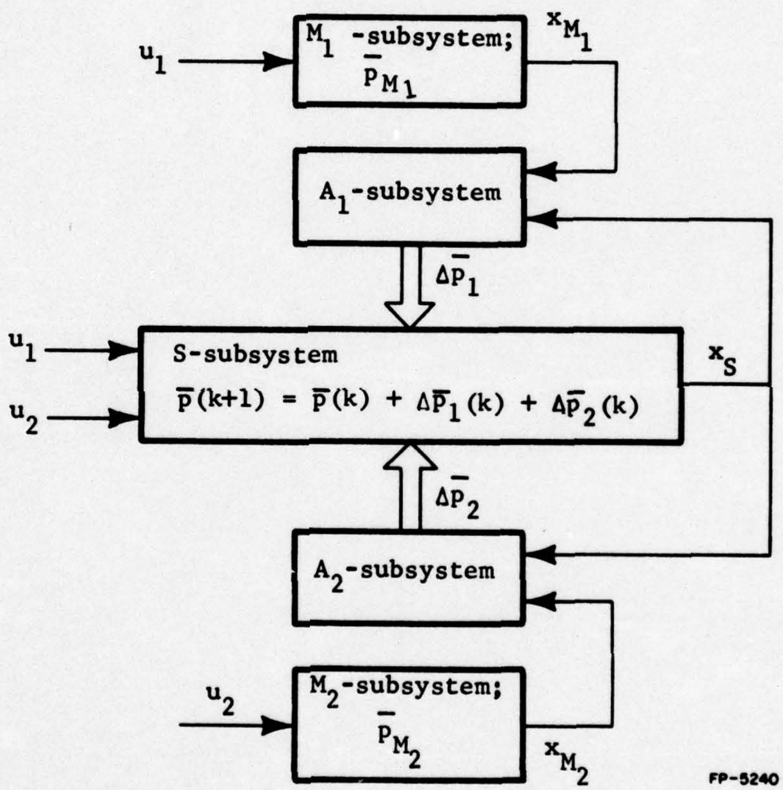

Figure 3.1-1 Two-Model MRAS Structure

where $e(k) = x_M(k) - x_S(k)$, then we say that (A_α, A_β) satisfy the partial ordering $A_\alpha \underset{U}{\geq} A_\beta$

The partial ordering for (M_1, M_2) identifies two distinct subclasses of 2M-MRAS, one subclass where the transient state response of one reference model to the origin is globally uniformly faster than for the other reference model, the other subclass where such a property does not hold. For example, when M_1 and M_2 are linear discrete-time systems, $M_1 \geq M_2$ implies that the maximum norm for all eigenvalues of the system matrix A_{M_1} is less than or equal to the minimum for all eigenvalues of A_{M_2}. A norm for a complex eigenvalue, $\lambda = \sigma + jw$, might be:

$$|\lambda| = (\sigma^2 + w^2)^{1/2} \tag{3.1-1}$$

The partial ordering for (A_α, A_β) provides a means for classifying the 2M-MRAS in terms of the relative "strengths" of the two adaptation algorithms A_α and A_β. For example, for the Landau adaptation algorithm (2.1-14), if two different adaptation gain matrices satisfy $F_\alpha(k) \geq F_\beta(k)$, it may be demonstrated through simulation that, when $U = \{u = \text{constant}\}$, $A_\alpha \underset{U}{\geq} A_\beta$. In Section 3.4 we will consider more specifically how perturbing the 2M-MRAS away from perfect symmetry will influence the equilibrium trajectories of the 2M-MRAS parameter vector. Both the effects of (A_1, A_2) and (M_1, M_2) will be considered.

We make one further distinction here between the 1M-MRAS and the 2M-MRAS. For the 1-model case we assumed incomplete knowledge about the plant (S-subsystem) parameters. For the 2-model case, it is assumed that plant parameters are known by both players; uncertainty here for each

player is restricted to the effect of the other player's reference model, adaptation algorithm, and external input on the S-subsystem state-trajectory.

For the remainder of this chapter we will assume that the M_i- and S-subsystems in Figure 3.1-1 are linear discrete-time, and the adaptation subsystems A_i are of the hyperstable type (with respect to the 1M-MRAS) introduced by Landau and treated in Chapter 2. Figure 3.1-2 depicts such a system. We define the following variables for this 2M-MRAS:

$$x_{M_i}^T(k) = [x_{M_i}^{oT}(k) \mid \bar{x}_{M_i}^{iT}(k)] = [y_{M_i}(k-n) \ldots y_{M_i}(k-1) \mid u_i(k-n) \ldots u_i(k)] \tag{3.1-2}$$

$$p_{M_i}^T = [a_{M_i}^T \mid \bar{b}_{M_i}^T] = [a_{M_{i_n}} \ldots a_{M_{i_1}} \mid b_{M_{i_n}} \ldots b_{M_{i_0}}] \tag{3.1-3}$$

$$\begin{aligned} \bar{x}_S^T(k) &= [x_S^{oT}(k) \mid \bar{x}_S^{1T}(k) \mid \bar{x}_S^{2T}(k)] \\ &= [y_S(k-n) \ldots y_S(k-1) \mid u_1(k-n) \ldots u_1(k) \mid u_2(k-n) \ldots u_2(k)] \end{aligned} \tag{3.1-4}$$

$$\begin{aligned} \bar{p}^T(k) &= [a^T(k) \mid \bar{b}^{1T}(k) \mid \bar{b}^{2T}(k)] \\ &= [a_n(k) \ldots a_1(k) \mid b_n^1(k) \ldots \mid b_n^2(k) \ldots b_0^2(k)] \end{aligned} \tag{3.1-5}$$

$$\bar{x}_{S_i}^T = [x_S^{oT}(k) \mid \bar{x}_S^{iT}(k)] \tag{3.1-6}$$

$$\bar{p}_i(k) = [a^T(k) \mid \bar{b}^{iT}(k)] \tag{3.1-7}$$

$$\epsilon_i(k) = y_{M_i}(k) - y_S(k) \tag{3.1-8}$$

$$e_i^T(k) = [\epsilon_i(k-n) \ldots \epsilon_i(k-1)] \tag{3.1-9}$$

$$v_i(k) = \epsilon_i(k) - a_{M_i}^T e_i(k). \tag{3.1-10}$$

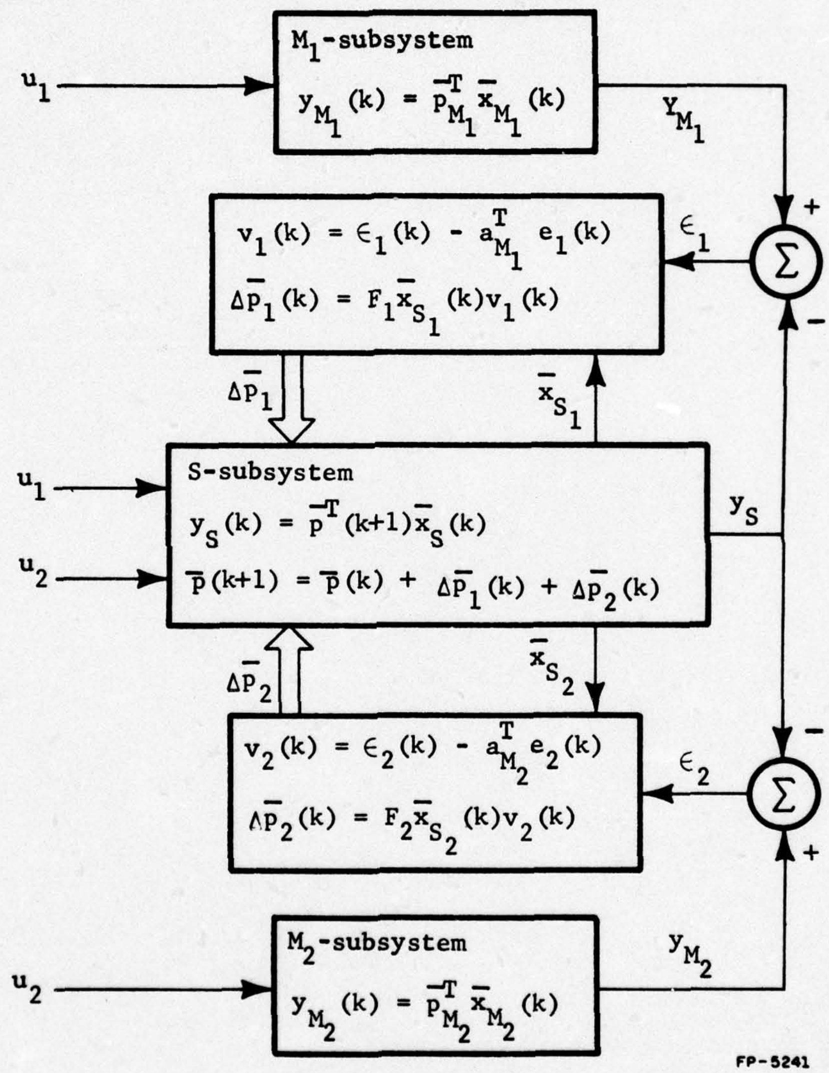

Figure 3.1-2 Two-Model MRAS with Landau Adaptation

The concept of the multi-model MRAS has been motivated by the theory of differential dynamic games. An n-player dynamic game may be defined by a plant model whose inputs are partitioned into n distinct groups, associated with distinct players, and a performance index associated with each player. (The players are also referred to as control agents or decision makers by some authors.) Each player desires to minimize his own performance index by implementing a control policy. However, since each player's performance is affected by the control decisions of the other players, all players cannot simultaneously (and independently) minimize their respective performance indices. Thus, unlike the (1-player) optimal control problem, it becomes expedient to view optimality in terms of a strategy adopted by each player, yielding strategy-dependent controls and system trajectories. The most characteristic strategies in game theory include Nash [57], Stackelberg [56], Pareto [57], and minimax [17].

In Chapter 1 we developed a relationship between the 1M-MRAS and a linear-quadratic optimal regulator problem. The M-subsystem reflected the desired performance of the controlled plant (S-subsystem), and the optimal regulator had an equivalent interpretation in the optimal control context for nominal plant parameters. Thus we chose the M-subsystem to be the optimal regulator obtained using nominal plant parameter values. In the 2M-MRAS case, each M-subsystem will be chosen equal to the regulator obtained by either optimizing that player's performance index, given some assumption about the control strategies chosen by the other player, or by solving a separate dynamic game based on assumed strategy and estimated performance index for the other player. We will now consider in detail how the (2M-MRAS, dynamic game) pair is developed.

Consider the linear-quadratic regulator game defined by:

$$x(k+1) = Ax(k) + B_1u_1(k) + B_2u_2(k) \tag{3.1-11}$$

$$J_i = \frac{1}{2}\sum_{k=0}^{\infty}(x^TQ_ix + 2u_i^TM_{ii}x + 2u_j^TM_{ij}x + 2u_i^TG_{ij}u_j + u_i^TR_{ii}u_i + u_j^TR_{ij}u_j), \tag{2.1-12}$$

$$(i,j) = \{(1,2),(2,1)\}.$$

Considering $u^T = [u_1^T \mid u_2^T]$, we rewrite (3.1-11) and (3.1-12) as:

$$x(k+1) = Ax(k) + Bu(k) \tag{3.1-13}$$

$$J_i = \frac{1}{2}\sum_{k=0}^{\infty}(x^TQ_ix + u^TM_ix + u^TR_iu), \quad i = 1,2 \tag{3.1-14}$$

where:

$$B = [B_1 \mid B_2]$$

$$M_1 = \begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} \qquad M_2 = \begin{bmatrix} M_{21} \\ M_{22} \end{bmatrix} \tag{3.1-15}$$

$$R_1 = \begin{bmatrix} R_{11} & G_{12} \\ G_{12}^T & R_{12} \end{bmatrix} \qquad R_2 = \begin{bmatrix} R_{21} & G_{21}^T \\ G_{21} & R_{22} \end{bmatrix}$$

Solutions for this dynamic game may be obtained by assuming that the two players jointly adopt one of the game strategies mentioned above. Now consider the 2M-MRAS with S-subsystem transfer function matrix defined by:

$$H_s(z) = [C^T(zI-A)^{-1}B_1 \vdots C^T(zI-A)^{-1}B_2]$$

where $\qquad C^T = [1 \quad 0 \; \ldots \; 0].$

The M_1- and M_2-subsystems must still be chosen. We consider several alternate viewpoints a player might take in assuming something about the other player's control strategy or performance index.

The most optimistic viewpoint by player (i) would be to assume player (j) is operating completely in the interest of player (i). M_i would

then be obtained by minimizing J_i with respect to both u_1 and u_2, or $u^T = [u_1^T \; u_2^T]$. Using (3.1-14), and assuming:

$$\begin{aligned} &Q_i \geq 0 \\ &R_i > 0 \\ &Q_i - M_i R_i^{-1} M_i \geq 0 \end{aligned} \tag{3.1-16}$$

the optimal control solution, using the variational approach [18] is:

$$u^*(k) = -R_i^{-1}\{M_i + B^T P_i H_i^{-1} W_i\} x(k) \tag{3.1-17}$$

where P_i is the unique positive-definite symmetric matrix satisfying:

$$P_i = Q_i - M_i^T R_i^{-1} M_i + W_i^T P_i H_i^{-1} W_i \tag{3.1-18}$$

and where:

$$H_i = I + BR_i^{-1} B^T P_i \tag{3.1-19}$$

$$W_i = A - BR_i^{-1} M_i \tag{3.1-20}$$

The closed-loop optimal plant matrix corresponding to (3.1-13), (3.1-17) is:

$$A_i^* = H_i^{-1} W_i \tag{3.1-21}$$

Thus the optimistic viewpoint would result in choosing M_i according to (3.1-21).

A less optimistic viewpoint which might be taken is for player (i) to estimate the performance index for player (j) and then to solve a dynamic game problem based on this estimate and an assumed strategy for both players. For example an assumed Nash strategy would yield a closed-loop regulator as defined by (3.3-9) in Section 3.3. This regulator would then be chosen by player (i) as his reference model M_i. Alternatively, player (i) might estimate directly the cost strategy to be chosen by player (j). For example, suppose player (i) assumes:

$$u_j = F_j x. \tag{3.1-22}$$

Then player (i) would solve the following optimization problem:

$$x(k+1) = \bar{A}_j x(k) + B_i u_i(k) \tag{3.1-23}$$

where:

$$\bar{A}_j = A + B_j F_j, \tag{3.1-24}$$

$$J_i = \frac{1}{2} \sum_{k=0}^{\infty} (x^T \bar{Q}_i x + 2u_i^T \bar{M}_{ii} x + u_i^T R_{ii} u_i) \tag{3.1-25}$$

where:

$$\bar{Q}_i = Q_i + 2F_j^T M_{ij} + F_j^T R_{ij} F_j \tag{3.1-26}$$

$$\bar{M}_{ii} = M_{ii} + G_{ij} F_j. \tag{3.1-27}$$

The solution to this problem takes the same form as (3.1-17)-(3.1-21), with appropriate substitutions. This viewpoint is similar to the philosophy inherent in the MRAS approach to control system design. For the 1M-MRAS, the uncertainty in plant parameters is dealt with by assuming nominal values in order to obtain a reference model from the optimization process. Then the adaptation loop is introduced to account for the fact that actual plant parameters may not be at the nominal values assumed. In the 2M-MRAS, M_i would be obtained by assuming a nominal u_j (in the regulator problem here, a state-feedback form). Player (i) would then assume that his adaptation algorithm compensated for the fact that the actual u_j might differ from the nominal u_j.

As a special case and most pessimistic example of this last viewpoint, player(i) would assume player(i) is operating to maximize player(i)'s cost J_i; i.e., a minimax strategy. Thus, player(i) would obtain u_j by maximizing J_i with respect to u_i. The resulting solution for player (i) would then be used to define the M_i-subsystem.

In Section 3.3 we consider in more detail the viewpoint where each player estimates the other player's control in order to obtain definitions for M_1 and M_2. We consider there the case where player(i) assumes $u_j = 0$. Then (3.1-23), (3.1-25) become

$$x(k+1) = Ax(k) + B_i u_i \tag{3.1-28}$$

$$J_i = \frac{1}{2} \sum_{k=0}^{\infty} (x^T Q_i x + 2u_i^T M_{ii} x + u_i^T R_{ii} u_i) \tag{3.1-29}$$

and M_i is obtained by minimizing (3.1-29) with respect to u_i. This resulting 2M-MRAS is then compared with the dynamic game where each player adopts the Nash strategy.

3.2 Stability Analysis for the 2M-MRAS

We shall consider the 2-input 1-output linear discrete-time 2M-MRAS of Figure 3.1-2 and (3.1-2)-(3.1-10), where A_1 and A_2 each correspond to a hyperstable Landau adaptation algorithm, studied in Chapter 2. The hyperstability of the 1-model case was established by transforming the MRAS structure to the Popov negative feedback form of Figure 2.1-3 consisting of a linear and nonlinear subsystem. The linear subsystem was required to be positive-real, and the nonlinear subsystem to satisfy a positivity inequality. Since the nonlinear subsystem could be viewed as a linear time-varying subsystem with respect to the input and output, Landau has used the time-varying version of the discrete positive-real lemma [35]

to establish positivity of the nonlinear subsystem. Thus, our first step in analyzing the stability of this 2M-MRAS is to obtain an equivalent Popov-type structure consisting of a linear subsystem and nonlinear subsystem interconnected by negative feedback.

By analogy with the 1-model MRAS, we obtain a Popov-type structure whose linear subsystem describes the dynamics for $e_1(k)$ and $e_2(k)$, (3.1-9), with outputs $v_1(k)$ and $v_2(k)$, (3.1-10), and whose nonlinear system describes the dynamic equations for parameter errors $\delta\bar{p}_1(k)$ and $\delta\bar{p}_2(k)$ under the assumed adaptation algorithm. From (3.1-8) we obtain:

$$\varepsilon_i(k) = a_{M_i}^T f_i(k) + \delta\bar{p}_i^T(k+1)\bar{x}_{S_i}(k) - \bar{b}_S^{j^T}(k+1)\bar{x}_S^j(k) \tag{3.2-1}$$

where $f_i(k) = x_{M_i}^o(k) - x_S^o(k)$

and (3.2-2)

$$\delta\bar{p}_i(k) = \bar{p}_{M_i} - \bar{p}_i(k)$$

Thus the dynamic equation for the state errors becomes

$$e_i(k+1) = I_n^1 e_i(k) + B^2\varepsilon_i(k), \tag{3.2-3}$$

where I_n^1 is defined in Appendix A, and $B^{2^T} = [0\ .\ .\ .\ 0\ \ 1]$.

<u>Lemma 3.2.1</u> The variable $f_i(k)$ in (3.2-2) is equal to $e_i(k)$ for all k, if $e_i(0) = f_i(0)$.

<u>Proof</u>: By expressing the M_i-subsystem and S-subsystem in the I-O Delay state-variable form (see (2.2-7) - (2.2-13)),

$$x_{M_i}(k+1) = \begin{bmatrix} A_{M_i} & \hat{A}_{M_i}(1) \\ 0 & I_n^1 \end{bmatrix} x_{M_i}(k) + \begin{bmatrix} B^1_{M_i} \\ B^2 \end{bmatrix} u_i(k)$$

$$x_S(k+1) = \begin{bmatrix} A_S(k+1) & \hat{A}^1_S(1,k+1) & \hat{A}^2_S(1,k+1) \\ 0 & I_n^1 & 0 \\ 0 & 0 & I_n^1 \end{bmatrix} x_S(k) + \begin{bmatrix} B^1_{S_1}(k+1) & B^1_{S_2}(k+1) \\ B^2 & 0 \\ 0 & B^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

we may obtain the dynamic equation for $f_i(k)$ as

$$\begin{aligned} f_i(k+1) &= A_{M_i} f_i(k) + [A_{M_i} - A_S(k+1)]x_S^o + [\hat{A}_{M_i}(1) - \hat{A}^i_S(1,k+1)]x^i_S \\ &\quad - \hat{A}^j_S(1,k+1)x^j_S + [B^1_{M_i} - B^1_{S_i}(k+1)]u_i - B^1_{S_j}(k+1)u_j \\ &= A_{M_i} f_i(k) + B^2\{\delta\overline{p}_i^T(k+1)\overline{x}_{S_i} - \overline{b}^{j^T}_S(k+1)\overline{x}^j_S\} \\ &= I_n^1 f_i(k) + B^2\{a_{M_i} f_i + \delta\overline{p}_i(k+1)\overline{x}_{S_i} - \overline{b}^{j^T}_S(k+1)\overline{x}^j_S\} \\ &= I_n^1 f_i(k) + B^2 \varepsilon_i(k) \end{aligned}$$

Thus, assuming $f_i(0) = e_i(0)$, (3.2-3) generates a sequence for $\varepsilon_i(k)$ which is identical to the sequence generated for $f_i(k)$. ■

Thus, if we assume $f_i(0) = e_i(0)$, and denoting

$$w_i(k) = \overline{x}^T_{S_i}(k)\delta\overline{p}_i(k+1) - \overline{x}^{j^T}_S(k)\overline{b}^j_S(k+1) \qquad (3.2\text{-}4)$$

we may rewrite (3.2-1), (3.2-3) and (3.1-10) as:

$$\varepsilon_i(k) = a_{M_i}^T e_i(k) + w_i(k) \tag{3.2-5}$$

$$e_i(k+1) = A_{M_i} e_i(k) + B^2 w_i(k) \tag{3.2-6}$$

$$v_i(k) = w_i(k) \tag{3.2-7}$$

The Landau adaptation algorithm (2.1-14) is repeated here for reference:

$$\overline{p}_i(k+1) = \overline{p}_i(k) + F_i \overline{x}_{S_i}(k) v_i(k) \tag{3.2-8}$$

where $\overline{p}_i$ is given by (3.1-7) and $F_i = F_i^T > 0$ is assumed constant here, rather than the more general time-varying case (2.1-15). Then

$$\delta\overline{p}_i(k+1) = \delta\overline{p}_i(k) - F_i \overline{x}_{S_i}(k) v_i(k) \tag{3.2-9}$$

Denote the nonlinear subsystem outputs by $z_i(k) = -w_i(k)$. Then:

$$z_i(k) = -x_{S_i}^T \delta\overline{p}_i(k+1) + \overline{x}_S^{j\,T} \overline{b}_S^j(k+1) \tag{3.2-10}$$

$$+ -x_{S_i}^T \delta p_i(k) + \overline{x}_{S_i}^T F_i \overline{x}_{S_i} v_i(k) + \overline{x}_{S_j}^T \tilde{F}_j \overline{x}_{S_j} v_j(k) + \overline{x}_S^{j\,T} \overline{b}_S^j(k)$$

where

$$\tilde{F}_j = \begin{bmatrix} 0 & 0 \\ F_j^{21} & F_j^{22} \end{bmatrix} \tag{3.2-11}$$

The combined nonlinear subsystem outputs may be written:

$$\begin{bmatrix} z_1(i) \\ \\ z_2(k) \end{bmatrix} = - \left[\begin{array}{c|c} \bar{x}_{S_1}^T & [0 \;\vdots\; \bar{x}_S^2]^T \\ \hline [0 \;\vdots\; \bar{x}_S^1]^T & \bar{x}_{S_2}^T \end{array}\right] \left[\begin{array}{c} \delta\bar{p}_1(k) \\ \hline \delta\bar{p}_2(k) \end{array}\right]$$

$$+ \left[\begin{array}{c|c} \bar{x}_{S_1}^T \; F_1 \; \bar{x}_{S_1} & \bar{x}_{S_2}^T \; \tilde{F}_2 \; \bar{x}_{S_2} \\ \hline \bar{x}_{S_1}^T \; \tilde{F}_1 \; \bar{x}_{S_1} & \bar{x}_{S_2}^T \; F_2 \; \bar{x}_{S_2} \end{array}\right] \left[\begin{array}{c} v_1(k) \\ \hline v_2(k) \end{array}\right] + \left[\begin{array}{c} \bar{x}_S^{2^T} \; \bar{b}_{M_2} \\ \hline \bar{x}_S^{1^T} \; \bar{b}_{M_1} \end{array}\right] \qquad (3.2\text{-}12)$$

Combining (3.2-6),(3.2-7), (3.2-9) and (3.2-12), the Popov structure for the 2M-MRAS is shown in Figure 3.2-1; (the time index k has been suppressed for convenience). The structure here is very similar to the 1-model case in Figure 2.1-3, with the exception here of an additional forcing term in the non-linear sybsystem. By choosing v_i as in (3.1-10), (i.e., letting the design vector $c_i = -a_{M_i}$), the transfer function matrix for the linear subsystem is simply:

$$H(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3.2\text{-}13)$$

which is clearly strictly positive-real. In order for the 2M-MRAS to be strictly asymptotically hyperstable, the nonlinear subsystem is assumed to satisfy the inequality:

$$\sum_{k=0}^{K} z^T(k)v(k) \geq -\gamma_o^2 \qquad \text{for all } K \geq 0 \qquad (3.2\text{-}14)$$
$$\gamma_0^2 > 0$$

With respect to $\begin{bmatrix} \delta\bar{p}_1 \\ \delta\bar{p}_2 \end{bmatrix}$ as state variable, v(k) as input, and z(k) as output,

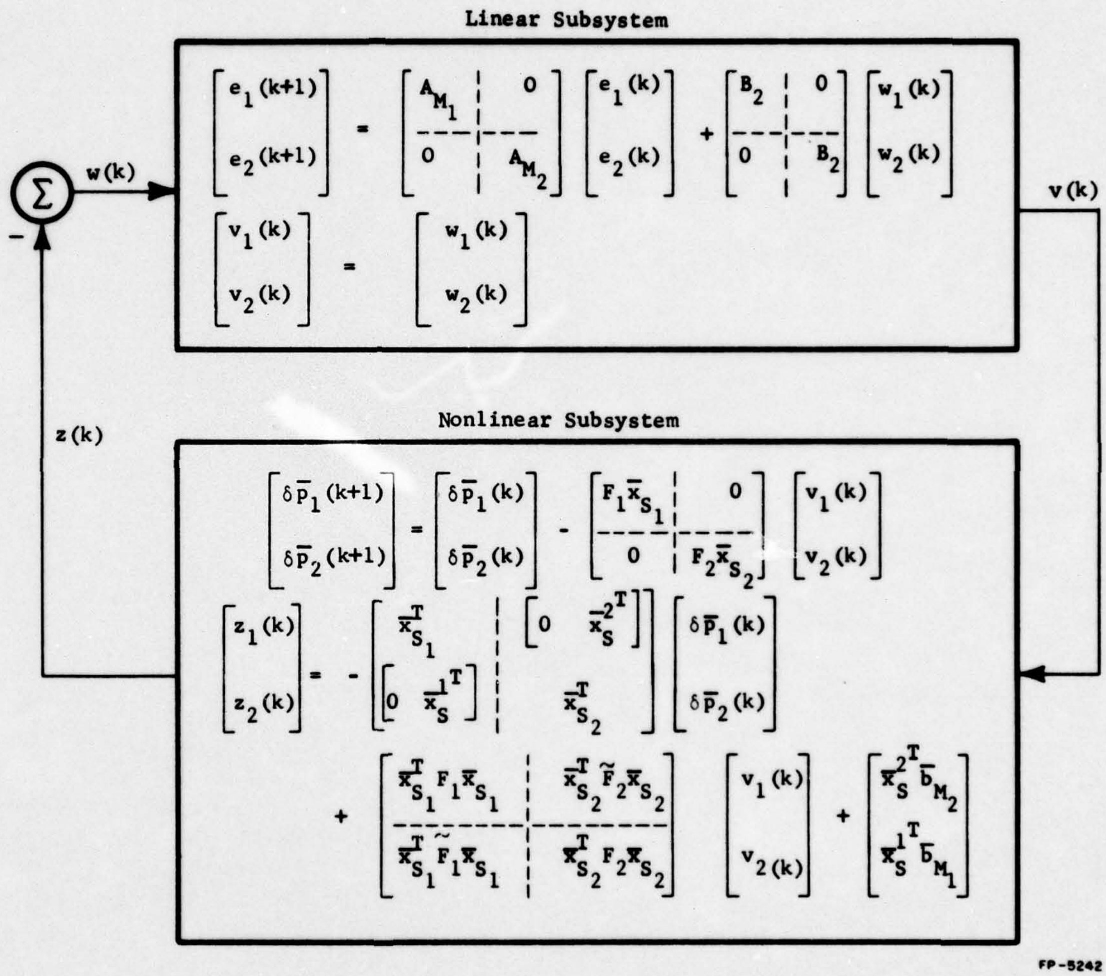

Figure 3.2-1 Popov Structure for 2M-MRAS

the nonlinear subsystem is actually linear time-varying, as in the 1-model case, but with the additional forcing term. If we neglect the forcing term appearing here, we might expect, due to its similarity to the 1-model case, that this linear time-varying system is also positive-real, i.e., it satisfies the time-varying version of the Discrete Positive Real Lemma (TVDPR). This subsystem is realized by:

$$\{A,B(k),C(k),D(k)\} = \left\{ I^{o}_{4n+2},\ -\left[\begin{array}{c|c} F_1\bar{x}_{S_1} & 0 \\ \hline 0 & F_2\bar{x}_{S_2} \end{array}\right],\ \left[\begin{array}{c|c} x^T_{S_1} & [0\ \ \bar{x}^{2^T}_S] \\ \hline [0\ \ \bar{x}^{1^T}_S] & \bar{x}^T_{S_2} \end{array}\right],\right.$$

$$\left.\left[\begin{array}{c|c} \bar{x}^T_{S_1} F_1 \bar{x}_{S_1} & \bar{x}^T_{S_2} \tilde{F}_2 \bar{x}_{S_2} \\ \hline \bar{x}^T_{S_1} \tilde{F}_1 \bar{x}_{S_1} & \bar{x}^T_{S_2} F_2 \bar{x}_{S_2} \end{array}\right] \right\} \tag{3.2-15}$$

We shall not prove that (3.2-15) satisfies the TVDPR Lemma; rather we shall assume it to be true in order to investigate how the additional forcing term in the nonlinear subsystem influences the positivity problem. Assuming (3.2-15) defines a positive system, a sufficient condition for the 2M-MRAS to be asymptotically hyperstable is:

$$[r_2(k) r_1(k)] \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} \geq 0 \text{ for all } k \geq 0 \tag{3.2-16}$$

where

$$r_i = \bar{x}^{i^T}_S\ \bar{b}_{M_i}.$$

This follows directly from (3.2-14). A set of sufficient conditions and a set of necessary and sufficient conditions for (3.2-16) to be true are given in the following proposition.

__Proposition 3.2.1__

Assume that (3.2-15) satisfies the TVDPR Lemma; then (a) sufficient conditions for which (3.2-16) is true are:

(r_1, r_2) lies within the region defined by the linear constraints:

$$\begin{array}{ll} \text{(a1)} & r_1 \geq 0,\; r_2 \geq \\ \text{(a2)} & r_1 \leq 0,\; r_2 \leq \end{array} \quad \frac{1+\gamma_1}{\gamma_1} r_1 - \frac{[\alpha_2+\gamma_1(\alpha_2-\alpha_1)]}{\gamma_1}$$

$$\begin{array}{ll} \text{(a3)} & r_2 \geq 0,\; r_1 \geq \\ \text{(a4)} & r_2 \leq 0,\; r_1 \leq \end{array} \quad \frac{1+\gamma_2}{\gamma_2} r_2 - \frac{[\alpha_1+\gamma_2(\alpha_1-\alpha_2)]}{\gamma_2}$$

where $\gamma_i = \bar{x}^T_{S_i} F_i \bar{x}_{S_i} \geq 0$

$$\alpha_i = [\bar{x}^T_{S_i} \quad [0 \quad \bar{x}^{j\,T}_S]] \begin{bmatrix} \delta\bar{p}_i \\ \delta\bar{p}_j \end{bmatrix}$$

and (b) a necessary and sufficient condition for which (3.2-16) is true is:

(r_1, r_2) lies within the elliptic region defined by:

$$(1+\gamma_1)r_1^2 + (1+\gamma_2)r_2^2 - (\gamma_1+\gamma_2)r_1 r_2$$

$$- [\alpha_2+\gamma_1(\alpha_2-\alpha_1)]r_1 - [\alpha_1+\gamma_2(\alpha_1-\alpha_2)]r_2 \leq 0$$

__Proof__

a) A sufficient condition for (3.2-16) to be true is that the separate terms $r_2 v_1 \geq 0$ and $r_1 v_2 \geq 0$ for all k. From (3.2-4) and (3.2-7), we obtain

$$\begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} (1+\gamma_2) - \gamma_2 & \alpha_1 - r_2 \\ \\ -\gamma_1 & (1+\gamma_1) \; \alpha_2 - r_1 \end{bmatrix}$$

where $\Delta = 1 + \gamma_1 + \gamma_2 > 0$

$$\text{Thus } r_2 v_1 = \frac{r_2\{\beta_2 r_1 - (1+\gamma_2) r_2 + [(1+\gamma_2)\alpha_1 - \beta_2\alpha_2]\}}{\Delta} \geq 0$$

$$r_1 v_2 = \frac{r_1\{-(1+\gamma_1) r_1 + \beta_1 r_2 + [(1+\gamma_1)\alpha_2 - \beta_1\alpha_1]\}}{\Delta} \geq 0$$

Since $\Delta > 0$, these two equations yield the sufficient conditions (a1) - (a4) directly.

b) By considering (3.2-16) directly, $r_1 v_2 + r_2 v_1 \geq 0$ becomes, using the definition above for v_1 and v_2:

$$\frac{\{-(1+\gamma_1) r_1^2 + \gamma_1 r_1 r_2 + [(1+\gamma_1)\alpha_2 - \gamma_1\alpha_1] r_1 - (1+\gamma_2) r_2^2 + \gamma_2 r_1 r_2 + [(1+\gamma_2)\alpha_1 - \gamma_2\alpha_2]\}}{\Delta} \geq 0$$

yielding the ellipse constraint directly. Since (b) is necessary and sufficient while (a) is only sufficient, the constraint region (a) is completely contained within the constraint region (b) in the (r_1, r_2)-plane. ∎

To illustrate the constraint conditions of this proposition, consider the following example which is valid for a single time instant k_o.

<u>Example 3.2.1</u>

Let $\gamma_1 = \gamma_2 = 1$, $\alpha_1 = \alpha_2 = 1$ at $k = k_o$. Then the linear constraints are:

a1) $r_1 \geq 0$, $r_2 \geq 2r_1 - 1$

a2) $r_1 \leq 0$, $r_2 \leq 2r_1-1$

a3) $r_2 \geq 0$ $r_1 \geq 2r_2-1$

a4) $r_2 \leq 0$ $r_1 \leq 2r_2-1$

and the elliptic constraint is:

b) $2r_1^2 + 2r_2^2 - 2r_1r_2 - r_1 - r_2 \leq 0$

The regions satisfying the constraints are shown in Figure 3.2-2. The interior double-hatched region corresponds to the sufficient conditions (a) while the circumscribed ellipse corresponds to the necessary and sufficient condition (b). ■

Although this proposition suggests the possibility that the 2M-MRAS may be made hyperstable by selecting the external input sequences, $\{u_1(k),u_2(k)\}$, appropriately, and thus $\{\bar{x}_S^1(k),\bar{x}_S^2(k)\}$, a straightforward method of doing so is not possible because the region constraining (r_1,r_2) depends on the unknown parameter error vectors, $\delta\bar{p}_1(k)$ and $\delta\bar{p}_2(k)$, imbedded in $\alpha_1(k)$ and $\alpha_2(k)$. Thus, the most conservative strategy is to choose $\bar{x}_S^1$ and $\bar{x}_S^2$ to maintain the values of r_1 and r_2 as close to the origin, as possible, since $(r_1,r_2) = (0,0)$ always lies in the constraint region, independent of α_1 and α_2. This strategy is possible since the model coefficients, $\bar{b}_{M_1}$ and $\bar{b}_{M_2}$ are assumed known by the respective players. However, we should remember that we are working under the (as yet unproved) assumption that the linear time-varying system is positive when $\frac{r_2}{r_1}$ is ignored.

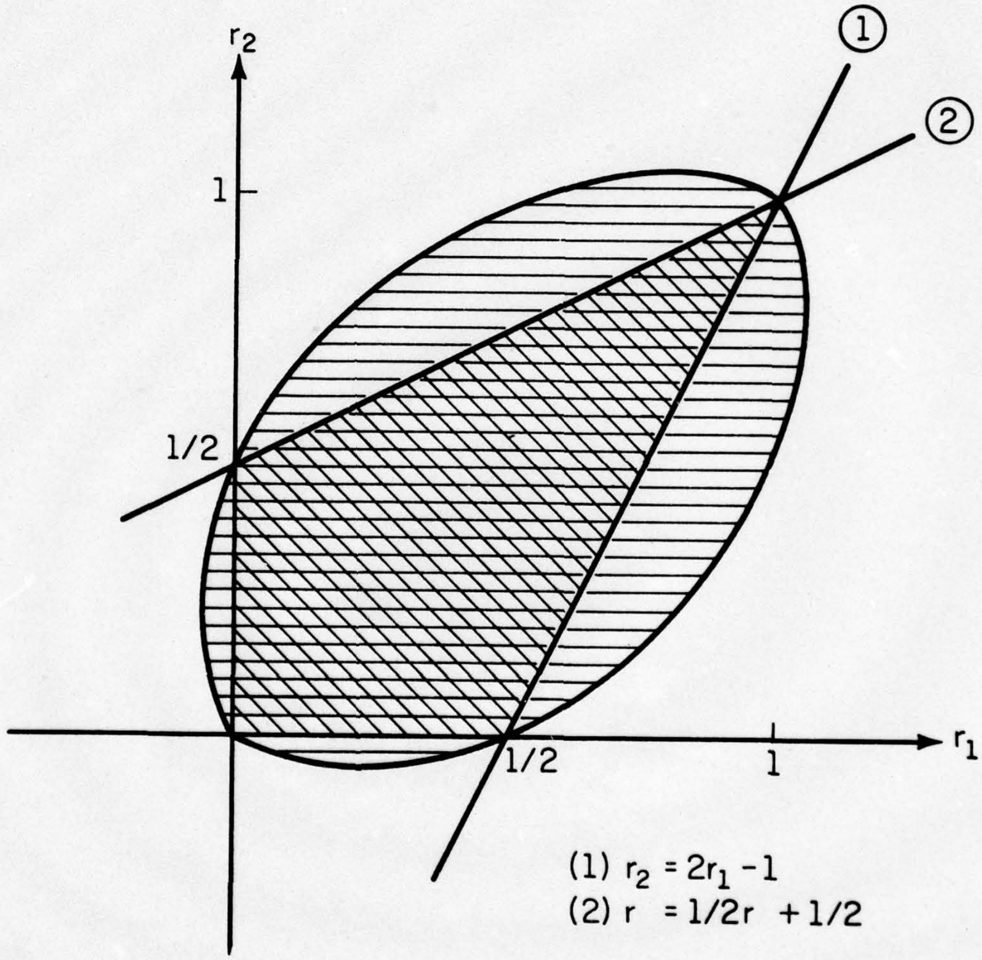

Figure 3.2-2 Example 3.2-1--Regions in (r_1, r_2)-plane Satisfying Proposition 3.2.1 Constraints

3.3 A Particular (2M-MRAS, Dynamic Game) Pair

In this section we will formulate a discrete-time dynamic game with quadratic performance indices and a 2M-MRAS according to the methods described in Section 3.1. Adopting a Nash strategy, the closed-loop Nash solution for the dynamic game will be stated, and then some Nash solution examples will be compared with the equilibrium parameter values of the corresponding 2M-MRAS structure. For this comparison we shall adopt the viewpoint, described in Section 3.1, that player(i) assumes a nominal function for player(j)'s control. Thus the reference models M_1 and M_2 are obtained by minimizing the performance indices J_1 and J_2 with respect to u_1 alone and u_2 alone, respectively. To simplify the minimization process we take player(j)'s assumed control to be $u_j = 0$. We may argue that this choice is no less reasonably than another if player(i) has no apriori knowledge about u_j.

The two-player dynamic game to be considered is as follows. The plant is defined by:

$$x(k+1) = Ax(k) + B_1u_1 + B_2u_2 \tag{3.3-1}$$

$$y(k) = C^Tx(k) + b_o^1u_1 + b_o^2u_2 \tag{3.3-2}$$

where A has the observable canonic form,

$$A = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ \vdots & 0 & & 0 & 1 \\ \vdots & & & & \\ a_n & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and $B_i^T = [b_1^i \; . \; . \; . \; b_n^i]$, $C^T = [1 \;\; 0 \; . \; . \; . \; 0]$.

The performance indices for the two players are given by:

$$J_i = \frac{1}{2} \sum_{k=0}^{\infty} (\overline{Q}_i y^2(k) + \overline{R}_{ii} u_i^2(k) + \overline{R}_{ij} u_j^2(k)), \quad i = 1,2 \quad \quad (3.3\text{-}3)$$

$$j = 1,2$$

Using (3.2-2), we obtain

$$J_i = \frac{1}{2} \sum_{k=0}^{\infty} (x^T Q_i x + 2u_i M_{ii} x + R_{ii} u_i^2 + 2u_i G_{ij} u_j + 2u_j M_{ij} x + R_{ij} u_j^2) \quad \quad (3.3\text{-}4)$$

where

$$\left.\begin{aligned} Q_i &= \overline{Q}_i C C^T \\ M_{ii} &= b_o^i \overline{Q}_i C^T \\ M_{ij} &= b_o^j \overline{Q}_i C^T \\ R_{ii} &= \overline{R}_{ii} + (b_o^i)^2 \overline{Q}_i \\ R_{ij} &= \overline{R}_{ij} + (b_o^j)^2 \overline{Q}_i \\ G_{ij} &= b_o^i b_o^j \overline{Q}_i \end{aligned}\right\} \quad \quad (3.3\text{-}5)$$

The plant parameters $a^T = [a_n \; . \; . \; . \; a_1]$, $\overline{b}_i^T = [b_n^i \; . \; . \; . \; b_o^i]$ in this problem correspond to the equivalent parameters in the 2M-MRAS S-subsystem. The closed-loop Nash solution for this dynamic game, using the variational approach, is [11]:

$$u_{i_N}(k) = \frac{-R_{ii}^{-1}}{\Delta} [M_i - G_{ij} R_{jj}^{-1} M_j] x(k) \quad \quad (3.3\text{-}6)$$

where

$$\left.\begin{aligned} \Delta &= \frac{R_{11} \, R_{22}}{R_{11} R_{22} - G_{12} G_{21}} \\ M_i &= M_{ii} + B_i^T K_i H^{-1} W \end{aligned}\right\}$$

$$\left.\begin{aligned} H &= I + \frac{1}{\Delta}\,[T_1R_{11}^{-1}B_1^TK_1 + T_2R_{22}^{-1}B_2^TK_2] \\ W &= A - \frac{1}{\Delta}\,[T_1R_{11}^{-1}M_{11} + T_2R_{22}^{-1}M_{22}] \\ T_i &= B_i - B_jR_{jj}^{-1}G_{ji} \end{aligned}\right\} \tag{3.3-7}$$

and where K_i is a matrix which satisfies the matrix equation:

$$\begin{aligned} K_i = Q_i &- \frac{1}{\Delta}\,\{M_{ii}^TR_{ii}^{-1}[M_i-G_{ij}R_{jj}^{-1}M_j] + M_{ij}^TR_{jj}^{-1}[-G_{ji}R_{ii}^{-1}M_i+M_j]\} \\ &- \frac{1}{\Delta}\,\{G_{ji}R_{ii}^{-1}M_i+M_j\}^TR_{jj}^{-1}\{M_{ij} - \frac{1}{\Delta}\,[R_{ij}R_{jj}^{-1}(-G_{ji}R_{ii}^{-1}M_i+M_j) \\ &+ G_{ij}R_{ii}^{-1}(M_i-G_{ij}R_{jj}^{-1}M_j) + B_j^TK_iH^{-1}W\} + A^TK_iH^{-1}W \end{aligned} \tag{3.3-8}$$

Sufficient conditions on A, B_i, Q_{ii}, R_{ii}, R_{ij}, M_{ii}, M_{ij}, and G_{ij} for existence of solutions to (3.3-8) are not known. From (3.3-6) the closed-loop system equation becomes:

$$x_N(k+1) = H^{-1}Wx_N(k) \tag{3.3-9}$$

Unlike the optimal control problem, the system matrix $H^{-1}W$ is not guaranteed to be asymptotically stable, so the solution is practical only in the case when this variational approach yields a stable system.

In order to specify reference models M_1 and M_2, we shall adopt the viewpoint, described earlier in this section, that player(i) assumes a nominal control $u_j = 0$, and then minimizes J_i with respect to u_i alone.

With these assumptions the optimization problem becomes

$$x(k+1) = Ax(k) + B_i u_i \tag{3.3-10}$$

$$J_i = \frac{1}{2} \sum_{k=0}^{\infty} (x^T Q_i x + 2u_i M_{ii} x + R_{ii} u_i^2) \tag{3.3-11}$$

where A, B_i, Q_i, M_{ii} and R_{ii} are as defined for the original dynamic game. The solution to this optimal control problem, using the variational approach [18] is:

$$u_i^* = -R_{ii}^{-1}[M_{ii}+B_i^T P_i H_i^{-1} W_i]x \tag{3.3-12}$$

where
$$H_i = I + B_i R_{ii}^{-1} B_i^T P_i \tag{3.3-13}$$

$$W_i = A - B_i R_i^{-1} M_i \tag{3.3-14}$$

and where P_i is the unique positive-definite symmetric matrix which satisfies the matrix equation:

$$P_i = Q_i - M_i^T R_i^{-1} M_i + W_i^T P_i H_i^{-1} W_i \tag{3.3-15}$$

The closed loop system equation becomes, from (3.3-10) and (3.3-12):

$$x^*(k+1) = H_i^{-1} W_i x^*(k) \tag{3.3-16}$$

and $H_i^{-1} W_i$ is guaranteed to yield an asymptotically stable system. These optimal systems (3.3-16) define the reference models M_1 and M_2 for the 2M-MRAS. We define the following 2M-MRAS restrictions so that the 2M-MRAS and the dynamic game have the same regulator-type structure. First of all, the parameter adaptation of the S-subsystem must be restricted to the a(k) portion of the parameter vector $\overline{p}(k)$ defined in (3.1-5). If adaptation of

the $\bar{b}^i(k)$ portions of $\bar{p}(k)$ is allowed, change to the performance index matrices M_{ii}, M_{ij}, R_{ii}, R_{ij}, and G_{ij} is implied due to changes in b_o^i. Further, the convergence of $\bar{b}^i(k)$ to some fixed point for the 2M-MRAS has no counterpart in the dynamic game plant equation. In the game context, the parameters (B_i, b_o^i) are assumed to be fixed known quantities. These same fixed values will therefore be assumed for the M_1-, M_2-, and S-subsystems. We will refer to this as the "pole adaptation only" restriction for the 2M-MRAS. The second restriction requires that the external inputs u_1 and u_2 of the 2M-MRAS equal zero throughout the MRAS operation. This corresponds to the fact that the dynamic game under consideration is assumed to be of the regulator-type.

We now present several examples where we compare the Nash solutions of the dynamic game with the 2M-MRAS equilibrium solution. The comparisons are done in terms of the Nash equilibrium plant poles and the 2M-MRAS S-subsystem equilibrium poles. This comparison is followed by an exact analytic solution for the S-subsystem equilibrium, in Proposition 3.3.1. The optimal solutions to define M_1 and M_2 and Nash solutions are found numerically by iteratively solving (3.3-15) and (3.3-8) until a fixed point is reached.

<u>Example 3.3.1</u>

Consider the 1^{st}-order dynamic system whose transfer function and state variable representation are given by:

$$H(z) = \left[\frac{1}{z+2} \;\vdots\; \frac{1}{z+2} \right]$$

$$x(k+1) = -2x(k) + u_1(k) + u_2(k)$$

$$y(k) = x(k)$$

The performance indices for each player are given by:

$$J_1 = \frac{1}{2} \sum_{k=0}^{\infty} (x^2(k) + u_1^2(k))$$

$$J_2 = \frac{1}{2} \sum_{k=0}^{\infty} (.001\ x^2(k) + u_2^2(k)).$$

In terms of plant poles, the two optimal solution (and M_i-subsystem) poles are found to be:

$$M_1 = -.382$$

$$M_2 = -.5$$

The closed-loop Nash solution yields:

$$\text{Nash} = -.3821$$

The 2M-MRAS was set up for pole adaptation only, $F_1 = F_2 = 1$, and I-O Delay state initial conditions:

$$x_{M_1}^T(0) = x_{M_2}^T(0) = [1, 0]$$

$$x_S^T(0) = [1, 0, 0]$$

The parameter $a_S(k)$ converged to $a_S^* = -.4413$, independent of the initial value $a_S(0) \in [-2,2]$. We observe that the Nash solution falls very close to the faster of the two optimal solutions. However, the 2M-MRAS equilibrium represents a compromise (in fact, an exact average) between the two reference model poles. ■

<u>Example 3.3.2</u>

Consider the $2^{\underline{nd}}$ order system whose transfer function and state variable representation are given by:

$$H(z) = \left[\frac{1}{(z-.99)^2} \;\middle|\; \frac{1}{(z-.99)^2} \right]$$

$$x(k+1) = \begin{bmatrix} 1.98 & 1 \\ -0.9801 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2(k)$$

$$y(k) = [1,0]x(k)$$

The performance indices for each player are given by:

$$J_1 = \frac{1}{2} \sum_{k=0}^{\infty} (x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + u_1^2)$$

$$J_2 = \frac{1}{2} \sum_{k=0}^{\infty} (x^T \begin{bmatrix} .001 & 0 \\ 0 & 0 \end{bmatrix} x + u_2^2)$$

In terms of plant poles, the two optimal solution (and M_i-subsystem) poles are found to be:

$$M_1 = 0.3728 \pm j\ 0.3004$$

$$M_2 = 0.8755 \pm j\ 0.1096$$

which corresponds to:

$$a_{M_1}^T = [-0.21,0.79]$$

$$a_{M_2}^T = [-0.81,\ 1.74]$$

The closed-loop Nash solution yields poles at:

$$\text{Nash} = 0.3728 \pm j\ .3003$$

The 2M-MRAS was set up for pole adaptation only, $F_1 = F_2 = \text{diag}(100,100)$, and the I-O Delay state initial conditions:

$$x_{M_1}^T(0) = x_{M_2}^T(0) = [0,1,00]$$

$$x_S^T(0) = [0,1,0,0,0,0]$$

The equilibrium poles for the S-subsystem depended on the initial pole values. However, in all cases, the equilibrium poles tended toward a compromise between the poles for the two reference models. For example, when the initial S-subsystem poles lie at (.5,.5), they move to final values ($0.674 \pm j\ 0.321$); initial values at (.9,.9) converge to ($0.653 \pm j\ 0.325$). This dependence on the initial values is believed to be due to the fact that the regulator-type 2M-MRAS allows only a short transient period during which adaptation may take place before the subsystem states and state-errors become too small to affect further parameter change. The stationary value a_S^* in each trial represents the extent to which $a_S(k)$ approached the true system equilibrium before adaptation was halted.

The S-subsystem equilibrium is in contrast again to the Nash solution which falls very close to the faster of the two optimal solutions. ■

The properties of the closed-loop Nash strategy equilibrium and the 2M-MRAS equilibrium in the above examples suggest a basic difference between the two problems. First we discuss the Nash solution property. Denote $\varepsilon_i = \overline{Q}_i/R_{ii}$ as a measure of the relative weight attached to the state for each player in the dynamic game as compared with a normalized weight of 1 applied to the control energy $u_i^2(k)$. Then player 1 (since $\varepsilon_1 > \varepsilon_2$) provides a greater proportion of the regulation control in the sense that his

Nash feedback gain is larger than that of player 2. Further, player 2 provides less control in the Nash equilibrium case than he would in the simple optimal control case, yielding an improved cost $J_{2_N} < J_2^*$. This is because player 1 is providing "free" control from the viewpoint of player 2, assisting player 2 to achieve his goal. In Example 3.3-1, the solution of (3.3-19) is $P_1 = 4.2361$ while the solution of (3.3-7) is $K_1 = 4.2277$, implying almost identical control efforts for player 1. In contrast $P_2 = 2.99$ while $K_2 = 0.004$, a significant reduction in control effort for player 2 in the Nash game as compared with the optimal control solution. In the limit, as $\varepsilon_1/\varepsilon_2$ approaches ∞, the Nash transient response approaches the optimal transient response of player 1. In our examples $\varepsilon_1/\varepsilon_2 = 1000$, which is effectively ∞ as far as this phenomenon is concerned.

The 2M-MRAS equilibrium, on the other hand, appears from the examples to represent a true compromise between the two optimal trajectories as specified by the reference models M_1 and M_2. The "cooperation" which appears in the Nash regulator solution does not occur here. The goal in this case, rather than taking the state to the origin as quickly as possible compromised by the necessary control effort, is to take the S-subsystem state to the origin along the exact trajectory specified by either M_1 or M_2. Thus, it is not surprising to find that the S-subsystem poles in our two examples reach an equilibrium lying in a region between the poles corresponding to M_1 and M_2. A more precise and quantitative description of the 2M-MRAS equilibrium is given by the proposition below. First we state several lemmas which will be used to prove the proposition.

<u>Lemma 3.3.1</u> Given the set of linearly independent vectors in R^n, $\{x_o, \ldots, x_{n-1}\}$, the set of diadic linear transformations $\{X_i = x_i x_i^T \mid i = 0, \ldots, n-1\}$ has the following properties:

1) $\{R_o, \ldots, R_{n-1}\}$ spans R^n

2) $\bigcap_{i=0}^{n-1} N_i = \{0\}$

where R_i denotes the range space of X_i and N_i denotes the null-space of X_i.

<u>Proof</u>:

1) The range space R of a diadic linear transformation $X = xx^T$ has dimension = 1, and $R = \{y \in R^n \mid y = \alpha x, \alpha \in R\}$. Given that $\{x_o, \ldots, x_{n-1}\}$ is linearly independent, and $\alpha_i x_i \in R_i$, where $\alpha_i \neq 0$ but otherwise arbitrary, then $\{\alpha_o x_o, \ldots, \alpha_{n-1} x_{n-1}\}$ is also a linearly independent set. Therefore $\{R_o, \ldots, R_{n-1}\}$ spans R^n.

2) The null-space N of a diadic transformation $X = xx^T$ has dimension n-1, and is equal to the hyperplane $N = \{y \in R^n \mid x^T y = 0\}$. If $y \in N_o, \ldots, N_{n-1}$, then

$$\begin{bmatrix} x_o^T \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1}^T \end{bmatrix} y = 0$$

But $\{x_o, \ldots, x_{n-1}\}$ is a linearly independent set, so $y = 0$ is the only vector which may satisfy this equality, and thus lie simultaneously in each N_i, that is, the intersection $\bigcap_{i=0}^{n-1} N_i$. ■

The next two lemmas deal with the linear independence properties of the state sequence $\{x_o, x_1, \ldots, x_k\}$ generated by the phase canonic matrix A. We separate R^n into two regions, denoted $G^+(A)$, $G^-(A)$, which are defined as follows:

$$G^+(A) = \{x \in R^n \mid (A,x) \text{ is a completely controllable pair}\}$$
$$G^-(A) = R^n - G^+(A) \tag{3.3-17}$$

<u>Lemma 3.3.2</u> The two regions $G^+(A)$ and $G^-(A)$ satisfy the following conditions:

a) $G^-(A)$ is A-invariant; i.e., $AG^-(A) \subset G^-(A)$

b) If A is nonsingular, $G^+(A)$ is A-invariant

c) If A is singular, $AG^+(A) \subset G^-(A)$

<u>Proof</u>:

a) $x_o \in G^-(A)$ implies $R_o = [x_o, Ax_o, \ldots, A^{n-1}x_o]$ is singular. $AR_o = [Ax_o, A^2x_o, \ldots, A^nx_o] = [x_1 Ax_1, \ldots, A^{n-1}x_1] = R_1$ is also singular. Therefore $x_1 \in G^-(A)$.

b) $x_o \in G^+(A)$ implies R_o is nonsingular; $R_1 = AR_o$ is also nonsingular; therefore $x_1 \in G^+(A)$

c) $x_o \in G^+(A)$ implies R_o is nonsingular, but $AR_o = R_1$ is singular. Therefore $x_1 \in G^-(A)$.

For the cyclic matrices under consideration (cyclic since A is in phase-canonic form), $G^+(A) \neq \phi$ (the empty set). We may characterize $G^-(A)$ by applying results due to Kalman and Heymann [23,14] to the Jordan Canonic

Form of A (which has only one Jordan block associated with each eigenvalue). The results are rephrased so as to be consistent with the development here.

Lemma 3.3.3 [23,14] Let A_J be a cyclic matrix in Jordan Canonic Form, with $A_J = \mathrm{diag}\,[J_1, \ldots, J_S]$, J_i the Jordan block associated with eigenvalue λ_i. Partition $x \in R^n$ to correspond with the Jordan blocks of A_J; $x^T = [x_1^T \;\; \ldots \;\; x_S^T]$. Then x is a generator of R^n (or equivalently, (A_J, x) is a completely controllable pair) if and only if each $x_{i_\ell} \neq 0$, where x_{i_ℓ} is the last element of x_i, $i = 1, \ldots, S$.

From this lemma we may characterize $G^-(A_J)$. In the coordinate system for the Jordan Form, $x \in G^-(A_J)$ must lie in a hyperplane defined as follows:

$$H_i = \{x \mid y_i^T x = 0;\; y_i^T = [0 \;.\;.\;.\; 0 \;\alpha\; 0 \;.\;.\;.\; 0],\; \alpha \neq 0\} \tag{3.3-18}$$

where α appears in the $i_\ell^{\underline{th}}$ position in y.

There are S such hyperplanes making up $G^-(A_J)$. Therefore

$$G^-(A_J) = \bigcup_{i=1}^{S} H_i \tag{3.3-19}$$

With respect to the coordinate system associated with the phase-canonic representation, A, the hyperplanes are simply rotated by a linear transformation to a different orientation in R^n.

From (3.3-18) and (3.3-19), "almost every" $x \in R^n$ lies in $G^+(A_J)$, in the sense that, if $x \in G^-(A_J)$, i.e., $x \in H_i$ for some $i = 1, \ldots, S$, then an arbitrarily small perturbation of x in any direction other than in the hyperplane H_i to $\tilde{x} = x + \Delta d$, $\Delta > 0$ $d \notin H_i$, places $\tilde{x} \in G^+(A_J)$.

<u>Proposition 3.3.1</u> If for the 2M-MRAS the following conditions hold:

a) $u_1 = u_2 = 0$,

b) pole adaptation only,

c) $F_1 = F_2 = F = F^T > 0$,

d) $a_S(k)$ converges at $k = k_o$ to a_S^* which corresponds to stable S-subsystem poles,

e) $\overset{o}{x}_S(k_o) \in G^+(A_S^*)$

then $a_S^* = \dfrac{a_{M_1} + a_{M_2}}{2}$

<u>Proof:</u> The adaptation equation is given by:

$$a_S(k+1) = a_S(k) + F\overset{o}{x}_S[v_1(k) + v_2(k)]$$

From Proposition 3.2.1, we obtain from the regulator case:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{1+2\gamma} \begin{bmatrix} (1+\gamma) & -\gamma \\ -\gamma & (1+\gamma) \end{bmatrix} \begin{bmatrix} \delta a_1^T \\ \delta a_2^T \end{bmatrix} \overset{o}{x}_S$$

where $\gamma = \overset{o}{x}{}_S^T F \overset{o}{x}_S$, $\delta a_i = a_{M_i} - a_S(k)$

Substituting into the adaptation equation,

$$a_S(k+1) = [\overset{o}{I}_n - \frac{2F\overset{o}{x}_S\overset{o}{x}{}_S^T}{1+2\gamma}]\, a_S(k) + \frac{F\overset{o}{x}_S\overset{o}{x}{}_S^T}{1+2\gamma}(a_{M_1} + a_{M_2})$$

When the stable equilibrium point is reached at stage k_o,

$$F\overset{o}{x}_S\overset{o}{x}{}_S^T(2a_S^* - (a_{M_1} + a_{M_2})) = 0 \quad \text{for all } k \geq k_o.$$

Since $F > 0$, $2a_S^* - (a_{M_1} + a_{M_2})$ lies in the (n-1) dimensional null-space N_k of $X_k = x_S^o(k)x_S^{o^T}(k)$ for all $k \geq k_o$. Assumption (e) implies $(A_S^*, x_S^o(k_o))$ is a completely controllable pair, and therefore $R_o = [x_S^o(k_o)A_S^*x_S^o(k_o) \ldots A_S^{*n-1}x_S^o(k_o)] = [x_S^o(k_o)x_S^o(k_o+1) \ldots x_S^o(k_o+n-1)]$ is nonsingular. Based on the characterization of $G^-(A)$ in (3.3-25), we claim that assumption (e) is not significantly restrictive.

Denoting $X_\ell = [x_S^o(k_o+1)x_S^{o^T}(k_o+1)]$, and using Lemma 3.3.1, $[2a_S^* - (a_{M_1} + a_{M_2})]$ must lie in the intersection $\bigcap_{l=0}^{n-1} N_l$. But since $\{x_S^o(k_o), \ldots x_S^o(k_o + n-1)\}$ is a linearly independent set, the intersection contains only the origin.

Therefore $2a_S^* - (a_{M_1} + a_{M_2}) = 0$, or $a_S^* = \dfrac{a_{M_1} + a_{M_2}}{2}$.

This proposition implies that the equilibrium S-subsystem poles may be obtained from the characteristic equation:

$$z^n - \frac{1}{2}\sum_{i=1}^{n}(a_{M_{1_i}} + a_{M_{2_i}})z^{n-i} = 0 \qquad (3.3\text{-}20)$$

3.4 Quasi-Symmetric Effect on 2M-MRAS Performance

The 2M-MRAS quasi-symmetric elements, namely the adaptation subsystems and reference model subsystems, may exhibit the partial ordering properties introduced in Section 3.1. For the 2M-MRAS using Landau adaptation, the notion of $A_\alpha > A_\beta$ corresponds to the condition where the adaptation gain matrices satisfy $F_\alpha > F_\beta$. In the 1M-MRAS context, we find that the convergence rate of the state-error to the origin is greater for F_α compared with F_β.

The partial ordering $M_1 > M_2$ indicates that every M_1 subsystem

eigenvalue has a smaller magnitude than the M_2 eigenvalue of least magnitude. Thus, the transient motion of $x_{M_1}(k)$ to its steady-state value is more rapid than the corresponding motion of $x_{M_2}(k)$, given that the initial conditions of both subsystems are equal.

In this section we analyze these partial ordering properties of the 2M-MRAS and their influence on its performance. We restrict our analysis to the regulator-type 2M-MRAS ($u_1 = u_2 = 0$), with pole adaptation only, and we make assumptions about the form of F_1 and F_2 as necessary to obtain results which most clearly illustrate the main ideas.

During operation of the regulator-type 2M-MRAS, we shall also assume that initial states $x^o_{M_1}(0) = x^o_{M_2}(0) = x^o_S(0)$, and that $x^o_{M_1}$, $x^o_{M_2}$, and x^o_S all converge to the origin. It is not necessarily true that x^o_S converges to the origin; the 2M-MRAS may become unstable for certain combinations of (F_1, F_2) and initial states, in which case $\| x^o_S \|$ becomes unbounded. Since only pole adaptation is assumed here, we are interested in the motion of the S-subsystem parameter vector $a_S(k)$. In subsequent equations, we will delete the time index when no confusion is possible.

Adaptation of the parameter vector a(k) is given by:

$$a_S(k+1) = a_S(k) + F_1 x^o_S v_1(k) + F_2 x^o_S v_2(k) \qquad (3.4\text{-}1)$$

From Proposition 3.2.1, recalling that $u_1 = u_2 = 0$ here, and denoting

$$\gamma_i(k) = x^{o^T}_S(k) F_i x^o_S(k) \qquad (3.4\text{-}2)$$

we may express v_1 and v_2 as:

$$\begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} = \frac{1}{1+\gamma_1+\gamma_2} \begin{bmatrix} (1+\gamma_2) & -\gamma_2 \\ \\ -\gamma_1 & (1+\gamma_1) \end{bmatrix} \begin{bmatrix} \delta a_1^T \\ \\ \delta a_2^T \end{bmatrix} x_S^o \tag{3.4-3}$$

Using (3.4-1) and (3.4-3) we obtain the following equivalent nonlinear description of the 2M-MRAS dynamics, where the state here is composed of $x_S^o(k)$ and $a_S(k)$:

$$x_S^o(k+1) = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & \dots & 0 & 1 \\ & & a^T(k+1) & \end{bmatrix} x_S^o(k) \tag{3.4-4}$$

$$a_S(k+1) = [I_n^o - \frac{(F_1+F_2)x_S^o x_S^{o^T}}{1+\gamma_1+\gamma_2}]\, a_S(k) \tag{3.4-5}$$

$$+ \frac{[(1+\gamma_2)F_1-\gamma_1F_2]x_S^o x_S^{o^T} a_{M_1} + [(1+\gamma_1)F_2-\gamma_2F_1]x_s^o x_S^{o^T} a_{M_2}}{1+\gamma_1+\gamma_2}$$

This parameter adaptation equation will provide the basis for analyzing the effect of the quasi-symmetric pairs (F_1,F_2) and (a_{M_1},a_{M_2}) on the performance of the 2M-MRAS. For the regulator-type 2M-MRAS, given that stability is maintained, the primary criterion of performance will be the equilibrium state for the 2M-MRAS. Since we assume $x_S^o(k)$ converges to the origin, we are interested only in the dependency of the equilibrium parameter vector a_S^* on (F_1,F_2) and (a_{M_1},a_{M_2}).

Define $E_1(k)$ and $E_2(k)$ as:

$$E_1(k) = (F_1+F_2)^{-1}[(1+\gamma_2F_1-\gamma_1F_2] \qquad (3.4\text{-}6)$$

$$E_2(k) = (F_1+F_2)^{-1}[(1+\gamma_1)F_2-\gamma_2F_1] \qquad (3.4\text{-}7)$$

The inverse exists since both F_1 and F_2 are positive-definite and symmetric, implying (F_1+F_2) is also. When parameter equilibrium is reached, $a_S(k+1) = a_S(k) = a_S^*$, and (3.4-5) becomes ($x_S^o(k)$ has not necessarily reached the origin):

$$[x_S^o x_S^{oT}]a_S^* = E_1 x_S^o x_S^{oT} a_{M_1} + E_2 x_S^o x_S^{oT} a_{M_2} \qquad (3.4\text{-}8)$$

This equation alone does not yield a unique solution for a_S^*. In order to obtain a unique solution in terms of (F_1,F_2) and (a_{M_1},a_{M_2}), we make the following assumption about the adaptation gain matrices:

$$F_i = \text{diag }[f_i]; \quad f_i > 0, \text{ scalar} \qquad (3.4\text{-}9)$$

Although this restriction eliminates both direct coupling among the elements of $a_S(k)$ and weighting of one element of $a_S(k)$ versus the others in (3.4-1) and (3.4-5), this special case of adaptation gain has been used in earlier periods of MRAS theory development [32], and is a reasonable choice when a sensitivity analysis has not been performed for the S-subsystem to determine how the parameters should be weighted to minimize the state-error convergence rate.

Given (3.4-9), E_1 and E_2 become:

$$\begin{aligned} E_1 &= \text{diag}\left[\frac{f_1}{f_1+f_2}\right] \\ E_2 &= \text{diag}\left[\frac{f_2}{f_1+f_2}\right] \end{aligned} \qquad (3.4\text{-}10)$$

and thus E_i and $x_S^o x_S^{o^T}$ commute;

$$E_i x_S^o x_S^{o^T} = x_S^o x_S^{o^T} E_i \tag{3.4-11}$$

Using (3.4-11), we may rewrite (3.4-8) as:

$$[x_S^o x_S^{o^T}]\ (a_S^* - E_1 a_{M_1} - E_2 a_{M_2}) = 0 \tag{3.4-12}$$

This equation must hold for each stage $k \geq k_o$, where k_o is the stage when $a_S(k) = a_S^*$. This leads to the following proposition, which specified how the quasi-symmetric elements affect the S-subsystem equilibrium.

<u>Proposition 3.4.1</u> If for the 2M-MRAS the following conditions hold:

a) $u_1 = u_2 = 0$,

b) pole adaptation only,

c) $a_s(k)$ converges at $k = k_o$ to a_S^* corresponding to stable S-subsystem poles,

d) $x_S^o(k_o) \in G^+(A_S^*)$, where $G^+(A_S^*)$ is defined by (3.3-23), then a_S^* satisfies

$$a_S^* = \frac{f_1 a_{M_1} + f_2 a_{M_2}}{f_1 + f_2}\ .$$

<u>Proof</u>: Following the development in Proposition 3.3.1, we claim that $[x_S^o(k_o) \ . \ . \ . \ x_S^o(k_o+n-1)]$ is nonsingular, and thus $(a_S^* - E_1 a_{M_1} - E_2 a_{M_2})$ must lie in the intersection of null spaces N_1 for the matrices $X_1 = [x_S^o(k_o+1) x_S^{o^T}(k_o+1)]$, $1 = 0, \ldots, n-1$.

But $\bigcap_{1=0}^{n-1} N_1 = \{0\}$, so

$$a_S^* = \frac{f_1 a_{M_1} + f_2 a_{M_2}}{f_1 + f_2} \qquad \blacksquare$$

For this special adaptation gain case, (3.4-9), Proposition 3.4.1 clearly indicates the effects of non-symmetry in (F_1, F_2):

$$\left.\begin{array}{lll} \text{For } f_1/f_2 = 1, & & a^* = \dfrac{a_{M_1} + a_{M_2}}{2} \\ \lim\limits_{f_1/f_2 \to \infty} & & a^* = a_{M_1} \\ \lim\limits_{f_1/f_2 \to 0} & & a^* = a_{M_2} \end{array}\right\} \tag{3.4-13}$$

Given $f_1 = f_2$, the effect of $M_1 > M_2$ on the S-subsystem eigenvalues corresponding to a_S^* is less clear. When $a_{M_1} = a_{M_2} = a_M$, $a_S^* = a_M$. However, for $M_1 > M_2$, the condition

$$a_S^* = \frac{a_{M_1} + a_{M_2}}{2}$$

does not necessarily place the eigenvalues of the S-subsystem in a region lying between those of the M_1 and M_2 subsystems. The S-subsystem eigenvalues may overlap into either the M_1 or M_2 region, depending on the relative separation of eigenvalues within each group, M_1 and M_2, respectively.

The following numeric example illustrates the results of Proposition 3.4.1 and compares theoretical results with results obtained from simulation.

Example 3.4.1

Consider the $2^{\underline{nd}}$-order regulator-type MRAS with $a_{M_1}^T = [-.81, 1.8]$, $a_{M_2}^T = [-.25, 1]$, corresponding to poles at (0.9, 0.9) and (0.5, 0.5), respectively. The S-subsystem is initialized at $a_S^T(0) = [-.01, .2]$, corresponding to poles at (0.1, 0.1). The following results are obtained:

f_1	f_2	a_S^*-theoretical	a_S^*-simulated	poles-theoretical	poles-simulated
100	100	[-.53, 1.4]	[-.516, 1.38]	0.7 ± j .2	.69 ± j .199
10000	100	[-.8044,1.792]	[-.7276,1.706]	.896 ± j .04	.853 ± j .006
100	10000	[-.2555,1.008]	[-.309,1.09	.504 ± j .0388	.546 ± j .107

When $f_1 = f_2$ the theoretical and simulated results agree quite well. However, in the other two cases, $F_1 > F_2$ and $F_1 < F_2$, the simulated poles do not move all the way out ($F_1 > F_2$) or all the way in ($F_1 < F_2$) to the theoretical pole values. It has been found that as f_1 and f_2 are both increased, it is possible to reduce this error. Thus, we conclude that this error between the simulated and theoretical values is due to the regulator property of the S-subsystem; adaptation is diminished in proportion to how close x_S^o is to its equilibrium point at the origin. Large f_1 and f_2 cause more of the adaptation to occur early in the trajectory of x_S^o before it becomes too close to zero, thus improving the comparison between theoretical and simulated results.

Future analysis of the 2M-MRAS should consider the more general case of adaptation gains $F_i = F_i^T > 0$. In addition, the cases where u_1 and u_2 are non-zero and where numerator adaptation is permitted should be studied.

CHAPTER 4

CONCLUSIONS

4.1 Review

In this thesis several topics involving a parallel MRAS have been investigated. We briefly summarize here the main results.

For the discrete-time single-input single-output MRAS, we introduced the Input-Output Delay state-variable representation for the M- and S-subsystems. Controllability conditions for this representation expressed in terms of the equivalent transfer function were found and used to produce an M-subsystem whose order is less than that of the S-subsystem. This was done in such a way that the Landau adaptation algorithm remains compatible with the reduced order subsystem; thus the MRAS retains its hyperstable property.

For the MRAS applied to parameter identification, we considered two separate problems. The first dealt with asymptotic stability of the parameter-error, the other with hyperstable design realization when the M-subsystem (plant) parameters are unknown.

The asymptotic stability of parameter-error is not inherent in the hyperstable design. We obtained several algebraic and geometric representations for the parameter-error equilibrium subspace. From these representations we were able to state necessary and sufficient conditions for asymptotic stability of the parameter-error, which means that the parameter-error equilibrium subspace contains only the origin. The practical use of this result is limited by the fact that the conditions depend on the unknown plant parameters.

The Landau adaptation design guaranteeing hyperstability is not directly realizable because the M-subsystem (plant) parameters are unknown.

We proposed a modified adaptation design which satisfies the positive-real hyperstability conditions asymptotically, assuming the MRAS remains stable. We suggested a strategy to investigate the stability properties of this modified algorithm. This investigation might find useful a test developed by Siljak for positive-realness on the unit circle.

Using results from the theory of variable structure systems (VSS), we have established the stability of a particular continuous-time MRAS structure which uses discontinuous adaptation. This MRAS was first shown to be equivalent to a general VSS which is designed to reject measurable disturbances, and then stability results for VSS were applied to the MRAS.

A new MRAS structure, one with multiple reference models, was introduced. Motivation for such a structure is found from dynamic game theory. We examine the stability properties for the two-model MRAS (2M-MRAS), given that the two adaptation algorithms are of the hyperstable form (with respect to the one-model MRAS). A 2M-MRAS and a dynamic game have been jointly developed. Alternate viewpoints in obtaining the M_i-subsystems of the 2M-MRAS have been described. A procedure for comparing the 2M-MRAS parameter equilibrium with the strategy-dependent dynamic game solution has been developed and illustrated by comparing a particular 2M-MRAS equilibrium with the closed-loop Nash solution of the dynamic game. We have also obtained the 2M-MRAS regulator parameter-equilibrium as a function of M_i-subsystem and A_i-subsystem parameters, for a particular adaptation gain structure.

4.2 Suggestions for Further Research

The result in Section 2.3, pertaining to the reduction in M-subsystem transfer function order, actually maintained the same system order for both the M- and S-subsystems from a state-variable viewpoint. However, one could construct a MRAS where the order-reduction for either the M- or S-subsystem is explicit in the state-variable formulation; that is, the difference in order is not achieved through cancellable transfer function poles and zeroes. To apply Landau's adaptation algorithm in this context, it would be necessary, first of all, to define a new state-error vector. The hyperstability of this MRAS formulation should be studied. Preliminary investigations suggest that the difficulties encountered in the stability analysis for this system are similar to some of the problems encountered in the analysis of the 2M-MRAS in Section 3.2.

The questions developed in Section 2.5 concerning the hyperstability of a modified form of the Landau MRAS, where the design vector c is chosen to be time varying:

$$c(k) = -a(k) \tag{4.2-1}$$

should be investigated further. Use of the Siljak test during simulations would be helpful in determining whether it is necessary at every stage k to satisfy the Popov positive-real condition.

The results in Chapter 3 dealing with the multi-model MRAS provide a basis from which generalizations could be made in several directions. The S-subsystem equilibrium parameter vector for the regulator-type 2M-MRAS could be investigated, assuming more general forms for the adaptation gain matrices. Further generalization could be attempted by assuming nonzero external inputs

(e.g., two different constant set-points), and considering full (both poles and zeroes) parameter vector adaptation.

REFERENCES

1. Anderson, B.D.O., "Adaptive Identification of Multiple-Input Multiple-Output Plants," Proceedings of the 1974 IEEE Conf. on Decision and Control, pp. 273-281

2. Åström, K., and P. Eykhoff, "System Identification - a survey," Automatica, vol. 7, 1971, pp. 123-162.

3. Athans, M., and Falb, P., Optimal Control, McGraw-Hill Book Co., New York, 1966.

4. Birtle, R. G., The Elements of Real Analysis, John Wiley & Sons, Inc., New York, 1964.

5. Birkhoff, G., and T. C. Bartee, Modern Applied Algebra, McGraw-Hill Book Co., New York, 1970.

6. Carroll, R. L., and D. P. Lindorff, "An Adaptive Observer for Single-Input Single-Output Linear Systems," IEEE Trans. on Aut. Contr., vol. AC-18, 1973, pp. 428-435.

7. Choe, H. H., and P. N. Nikiforuk, "A Note on Lyapunov-type Controllers," IEEE Trans. on Aut. Contr., vol. AC-15, 1970, pp. 701-702.

8. Devaud, F. M., and J. Y. Caron, "Asymptotic Stability of Model Reference Systems with Bang-Bang Control," IEEE Trans. on Aut. Contr., vol. AC-20, 1975, pp. 694-696.

9. Edwards, V. E., and W. L. Garrard, "A Comparative Study of Two Model-Reference Adaptive Aircraft Control Systems," Proc. 1974 JACC Conf., pp. 33-44.

10. Gantmacher, F. R., The Theory of Matrices, vol.I, Chelsea Publishing Co., New York, 1959.

11. Gardner, B. F., "Discrete-Time Dynamic Games," Report R-691, Coordinated Sciences Laboratory, Univ. of Illinois at Urbana-Champaign, September 1975.

12. Grayson, L. P., "The Status of Synthesis Using Lyapunov's Method," Automatica, vol. 3, 1965, pp. 91-121.

13. Greub, W. H., Linear Algebra, 3rd Ed., 1967, Springer-Verlag, New York.

14. Heymann, M., "On the Input and Output Reducibility of Multivariable Linear Systems," IEEE Trans. on Aut. Contr., vol. AC-15, 1970, pp. 563-569.

15. Hirsch, J. J., and P. Peltie, "Real-Time Identification using Adaptive Discrete Models," Proc. 3rd IFAC Symposium on Sensitivity, Adaptivity, and Optimality, 1973, pp. 290-297.

16. Hitz L., and B.D.O. Anderson, "Discrete Positive-Real Functions and their Application to System Stability," Proc. IEE, vol. 116, 1969, pp. 153-155.

17. Ho, Y. C., A. E. Bryson, and S. Baron, "Differential Games and Optimal Pursuit-Evasion Strategies," IEEE Trans. on Aut Contr., vol. AC-10, 1965, pp. 385-389.

18. Holtzman, "Convexity and the Maximum Principle for Discrete Systems," IEEE Trans on Aut. Contr., vol AC-11, 1966, p. 30-35.

19. Ionescu, T., and R. Monopoli, "Discrete Model Reference Adaptive Control with an Augmented Error Signal," Proc. IFAC 6th Triennial World Congress, 1975, Preprints Part 1D, 50.5, pp. 1-9.

20. Isaacs, R., Differential Games, John Wiley and Sons, Inc., N. Y., 1965.

21. Johnson, C. D., and W. M. Wonham, "Another Note on the Transformation to Canonical (Phase-Variable) Form," IEEE Trans. on Aut Contr., vol. AC-11, 1966, pp. 609-610.

22. Kalman, R., "Mathematical Description of Linear Dynamic Systems," J. SIAM Control, Ser. A, vol 1, no. 2, 1963, pp. 152-192.

23. Kalman, R. E., "Irreducible Realizations and the Degree of a Matrix of Rational Functions," SIAM J. Control, vol 13, 1965, pp. 520-544.

24. Kim, C. H., and D. P. Lindorff, "Input Frequency Requirements for Identification Through Liapunov Methods," Int. J. Control, vol. 20, 1974, pp. 35-48.

25. Kokotovic, P. V., Medanic, J. V., Viskovic, MI., and S. P. Bingulac, "Sensitivity Methods in the Experimental Design of Adaptive Control Systems," 3rd IFAC Congress, London, 1966, pp. 45B1-12

26. Kostyleva, N. E., "Recovery of Differential Equations of a Plant by Methods of VSS," Automation and Remote Control, vol. 32, 1971, pp. 1588-1595.

27. Krutova, I. N., and V. Y. Rutkovskii,"Investigation of the Dynamics of MRAS for a Plant with Nonlinear Characteristics and Variable Parameters," Automation and Remote Control, vol. 25, 1964, pp. 795-802.

28. Krutova, I. N., and V. Y. Rutkovskii,"Analysis of a Second Order Model Reference Self-Adaptive System," Automation and Remote Control, vol. 26, 1965, pp. 73-86.

29. Kudva, P., and K. S. Narendra, "An Identification Procedure for Discrete Multivariable Systems," IEEE Trans. on Aut. Contr., vol. AC-19, 1974, pp. 549-552.

30. Kudva, P., and K. S. Narendra, "The Discrete Adaptive Observer," Proc. 1974 IEEE Conf. on Decision and Control, pp. 307-312.

31. Landau, I. D., "A Hyperstable Criterian for Model Reference Adaptive Control Systems," IEEE Trans. on Aut. Contr., vol. AC-14, 1969, pp. 552-555.

32. Landau, I. D., "Synthesis of Discrete Model Reference Adaptive Systems," IEEE Trans. on Aut. Contr., vol. AC-16, 1971, pp. 507-508.

33. Landau, I. D., "A Generalization of the Hyperstability Conditions for Model Reference Adaptive Systems," IEEE Trans. on Aut. Contr., vol. AC-17, 1972, pp. 246-247.

34. Landau, I. D., "A Survey of Model Reference Adaptive Techniques--Theory and Applications," Automatica, vol. 10, 1974, pp. 353-379.

35. Landau, I. D., "An Asymptotic Unbiased Recursive Identifier for Linear Systems," Proc. of the 1974 IEEE Conf. on Decision and Control, pp. 288-294.

36. Landau, I. D., "Unbiased Recursive Identification Using Model Reference Adaptive Techniques," IEEE Trans. on Aut. Contr., vol. AC-21, 1976, pp. 194-202.

37. LaSalle, J., and S. Lefschetz, Stability by Lyapunov's Direct Method with Applications, Academic Press, New York, 1961.

38. Lindorff, D. P., "Adaptive Control of Multivariable Systems," Proc. 3rd Asilomar Conf. Circuits and Systems, 1969, pp. 119-124.

39. Lindorff, D. P., and R. L. Carroll, "Survey of Adaptive Control Using Lyapunov Design," Int. J. Control, vol. 18, 1973, pp. 897-914.

40. Lion, P. M., "Rapid Identification of Linear and Nonlinear Systems," AIAA Journal, vol. 5, 1967, pp. 1835-1842.

41. Luenberger, D., "Observers for Multivariable Systems," IEEE Trans. on Aut. Contr., vol. AC-11, 1966, pp. 190-197.

42. Luenberger, D. G., "Canonical Forms for Linear Multivariable Systems," IEEE Trans. on Aut. Contr., vol. AC-12, 1967, pp. 290-293.

43. Luenberger, D. G., Optimization by Vector Space Methods, John Wiley & Sons, Inc., New York, 1969.

44. Luenberger, D., Introduction to Linear and Nonlinear Programming, Addison-Wesley Publishing Co., Reading, Mass., 1973.

45. Monopoli, R. V., "Synthesis Techniques Employing the Direct Method," IEEE Trans. on Aut. Contr., vol. AC-10, 1965, pp. 369-370.

46. Monopoli, R. V., "Model Reference Adaptive Control with an Augmented Error Signal," IEEE Trans. on Aut. Contr., vol. AC-19, 1974, pp. 474-483.

47. Narendra, K. S., and P. Kudva, "Stable Adaptive Schemes for System Identification and Control - Part I, II," IEEE Trans. on Systems, Man, and Cybernetics, vol. SMC-4, 1974, pp. 542-560.

48. Parks, P. C., "Lyapunov Redesign of Model Reference Adaptive Control Systems," IEEE Trans. on Aut. Contr., vol. AC-11, 1966, pp. 362-367.

49. Pearson, A. E., and V. R. Vanguri, "A Synthesis Procedure for Parameter Adaptive Control Systems," IEEE Trans. on Aut. Contr., vol. AC-16, 1971 pp. 440-449.

50. Popov, V. M., "The Solution of a New Stability Problem for Controlled Systems," Automation and Remote Control, vol. 24, 1963, pp. 1-23.

51. Popov, V. M., Hyperstability of Control Systems, Springer-Verlag, Berlin, 1973.

52. Sakharov, M. P., "Simplification of Sensitivity Models in the Design of Self-Adjusting and Adaptive Systems," Aut. Remote Contr., vol. 29, 1968, pp. 743-747.

53. Serdyukov, V. A., "Synthesis of Adaptive Systems by Lyapunov's Direct Method," Automation and Remote Control, vol. 33, 1972, pp. 1138-1144.

54. Shackcloth, B., and R. L. Butchart, "Synthesis of Model Reference Adaptive Systems by Lyapunov's Second Method," 2nd IFAC Symposium on Theory of Self-Adaptive Systems, Teddington, England, 1965, pp. 145-152.

55. Siljak, D. D., "Algebraic Criteria for Positive Realness Relative to the Unit Circle," J. Franklin Institute, vol. 296, 1973, pp. 115-112.

56. Stackelberg, H. von, The Theory of the Market Economy, Alan T. Peacock, trans., Oxford Univ. Press, 1952.

57. Starr, A. W. and Y. C. Ho, "Nonzero Sum Differential Games," J. of Optimization Theory and Applic., vol. 3, 1969, pp. 207-219.

58. Unbehauen, H., and Chr. Schmid, "Status and Industrial Application of Adaptive Control Systems," Automatic Control Theory and Applications, vol. 3, 1975, pp. 1-12.

59. Utkin, V. I., "Quasi-Invariant Control of Linear Plants with Disturbances," Engineering Cybernetics, No. 5, 1966, pp. 149-155.

60. Utkin, V., "Variable Structure Systems-Survey," Report T-27, Coordinated Sciences Laboratory, Univ. of Illinois at Urbana-Champaign, June 1976.

61. Van Valkenburg, M. E., Network Analysis, 2nd ed., Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.

62. Whitaker, H. P., Proceedings of the Self-Adaptive Flight Control Systems Symposium, WADC TR-59-49, March, 1959.

63. Winsor, C. A., and R. J. Roy, "Design of Model Reference Adaptive Control by Lyapunov Second Method," IEEE Trans. on Aut. Contr., vol. AC-13, 1968, p. 204.

64. Young, K. K., "Asymptotic Stability of Model Reference Systems with Variable Structure Control," submitted for publication to IEEE Trans. on Aut. Contr.

65. Zemlyakov, S. D., and V. Y. Rutkovskii, "Synthesizing Self-Adjusting Control Systems with a Standard Model," Automation and Remote Control, vol. 27, 1966, pp. 409-414.

66. Zemlyakov, S. D., and V. Y. Rutkovskii, "Generalized Adaptation Algorithms for a Class of Searchless Self-Adjusting Systems with a Model," Automation and Remote Control, vol. 28, 1967, pp. 935-940.

APPENDIX A

In this appendix we present some matrix notation and definitions, and then develop some algebraic propositions related to these matrices. The material here supports the developments of Chapter 2, serving primarily as a central repository for matrix notation, allowing the main text to flow more smoothly. The propositions are also of some interest in themselves in relation to the theory of linear algebra, and in the insight gained into the structure of certain matrix forms.

A.1 Algebraic Structure of Two Finite Matrix Sets

In this section we present our notation for two finite matrix sets, and then describe their algebraic properties under the binary operation, matrix multiplication. The first set, denoted I_N, consists of those N x N matrices with 1's on a single diagonal, and 0's elsewhere. The identity matrix, I_N^o, is a familiar element of this set. The second set, denoted P_N, consists of those N x N matrices with 1's on a single anti-diagonal, and zeroes elsewhere.

Definition A.1-1

$I_N = \{I_N^k | k \in \text{Integers}\}$, where the matrix elements of I_N^k satisfy:

$$I_N^k(k,j) = \begin{cases} 1 & \text{if } (j-i) = k \\ 0 & \text{otherwise} \end{cases}$$

Definition A.1-2

$P_N = \{P_N^k | k \in \text{Integers}\}$, where matrix elements of P_N^k satisfy

$$P_N^k(i,j) = \begin{cases} 1 & \text{if } (N+1)-(i+j) = k \\ 0 & \text{otherwise} \end{cases}$$

Thus, elements of I_N and P_N are of the form:

$$I_N^k = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & 1 & & \vdots \\ \vdots & & & & & & 0 \\ \vdots & & & & & & 1 \\ & & & & & & 0 \\ & & & & & & \vdots \\ 0 & & \cdots & & & & 0 \end{bmatrix} \quad P_N^k = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 1 & & & & \vdots \\ 0 & & & & & & \vdots \\ 1 & & & & & & \vdots \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & \cdots & & & 0 \end{bmatrix}$$

If $k > 0$, 1's appear on a superdiagonal or anti-superdiagonal.

If $k < 0$, 1's appear on a subdiagonal or anti-subdiagonal.

If $k = 0$, 1's appear on the main diagonal or main anti-diagonal.

The two sets, I_N and P_N, satisfy the following algebraic properties:

<u>Property 1</u>: $I_N^k = 0$ and $P_N^k = 0$ for all $|k| \geq N$. This follows since both $i, j \in [1,N]$; thus:

$$-(N-1) \leq (j-i) \leq (N-1)$$

$$-(N-1) \leq (N+1) - (i+j) \leq (N-1)$$

That is, the conditions for which a matrix element equals 1 can never be met.

<u>Property 2</u>:

a) $P_N^o \, I_N^k = P_N^{-k}$

b) $I_N^k \, P_N^o = P_N^k$

c) $P_N^o P_N^k = I_N^{-k}$

d) $P_N^k P_N^o = I_N^k$

e) $P_N^o I_N^k P_N^o = I_N^{-k}$

f) $P_N^o P_N^k P_N^o = P_N^{-k}$

g) $P_n^{k_1} P_N^{k_2} = I_N^{k_1} I_N^{-k_2}$

h) $P_N^{k_1} I_N^{k_2} = I_N^{k_1} P_N^{-k_2}$

i) $I_N^{k_2} P_N^{k_1} = P_N^{k_2} I_N^{-k_1}$

<u>Property 3:</u>

a) $I_N^{k_1} I_N^{k_2} = I_N^{(k_1+k_2)}$ if $\mathrm{sgn}(k_1 \cdot k_2) = +1$

b) $I_N^{k_1} I_N^{k_2} + I_N^{k_2} I_N^{k_1} = I_N^{(k_1+k_2)}$ if $\mathrm{sgn}(k_1 \cdot k_2) = -1$

<u>Property 4</u>:

a) $P_N^{k_1} P_N^{k_2} = I_N^{(k_1-k_2)}$ if $\mathrm{sgn}\ (k_1 \cdot k_2) = -1$

b) $P_N^{k_1} P_N^{k_2} + P_N^{-k_2} P_N^{-k_1} = I_N^{(k_1-k_2)}$ if $\mathrm{sgn}\ (k_1 \cdot k_2) = +1$

This follows from properties 2(g) and 3.

Properties 1 and 3(a) imply that I_N^k has a nilpotency index [10] equal to [N/k], where the notation

$$n = [N/k] = \underset{\text{Integers}}{\text{Min}} \{n \mid n \geq N/k\}$$

The elements of I_N and P_N occur in the development of Chapter 2 as either right- or left-hand linear operators on general matrices or vectors of

compatible dimension. We include a qualitative description of their effects on general matrices here to clarify that development.

Given $A \quad R^{NxN}$,

1) $AI_N^{\pm k}$ shifts columns of A right (+) or left (-) k steps, filling with 0-columns.

2) $I_N^{\pm k}A$ shifts rows of A up (+) or down (-) k steps, filling with 0-rows.

3) $AP_N^{\pm k}$ pivots A about its central vertical axis, then performs $AI_N^{\pm k}$.

4) $P_N^{\pm k}A$ pivots A about its central horizontal axis, then performs $I_N^{\pm k}A$.

$S_N = I_N \quad P_N \quad \{0\}$, and the binary operation, matrix multiplication, denoted $\cdot$, then $\{S_N, \cdot\}$ does <u>not</u> form a binary algebra or semigroup [5]. Properties 3 and 4 show that $\cdot$ is not a closed operation; i.e., it generates elements which are not members of S_N.

A.2 Matrix Relationships Involving the Companion Form

The companion form for a matrix appears often in problems dealing with state-variable realizations of a LTI system [21,42], and is also a fundamental form in studying the general theory of linear transformations [10]. In this section we develop some notation and relations which involve the companion form, and which also appear in Chapter 2.

A.2.1 The Output Delay System

Consider the difference equation:

$$y(k) = \sum_{i=1}^{N} a_i y(k-i) \tag{A.2-1}$$

A state-variable realization which occurs naturally from this equation is:

$$w(k+1) = F\, w(k) \tag{A.2-2}$$

$$y(k) = a^T w(k) \tag{A.2-3}$$

where $w^T(k) = [y(k-N) . . . y(k-1)]$ (A.2-4)

$a^T = [a_N \, . \, . \, . \, a_1]$ (A.2-5)

$$F = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 1 \\ a_N & \cdots & & a_1 \end{bmatrix} = I_N^1 + \begin{bmatrix} 0 & & 0 \\ & & \\ 0 & & 0 \\ a_N & \cdots & a_1 \end{bmatrix} \tag{A.2-6}$$

The dynamic system matrix is known as the companion or $1^{\underline{st}}$ natural form [10], and is a fundamental canonic matrix in the study of linear transformations. We will refer to the state realization w(k) as the <u>Output Delay State Realization</u>.

The next proposition provides the basis for stating the necessary and sufficient conditions for observability of the pair (a^T,F). It also provides insight into the algebraic structure of the companion matrix F.

<u>Proposition A.2.1</u>

a) F has an eigenvalue $\lambda = 0$ of multiplicity (N-k) and

b) $\rho\{F^{\alpha}\} = k$ for all $\alpha \geq (N-k)$ if and only if the elements of $a^T = [a_N \ . \ . \ . \ a_1]$ satisfy:

$$(a_N, \ . \ . \ . \ a_{k+1}) = 0; \ a_k \neq 0; \quad (a_{k-1}, \ . \ . \ . \ a_1) \text{ arbitrary}$$

where k may take the value $k \in [0,N]$, and $\rho\{\cdot\}$ denotes the matrix rank.

Proof: Since F is cyclic [10], the characteristic and minimal polynomials, $\Delta(\lambda)$ and $\psi(\lambda)$, respectively, are equal:

$$\Delta(\lambda) = \psi(\lambda) = \lambda^{N-k}(\lambda^k - \sum_{i=1}^{k} a_i \lambda^{k-i})$$

Thus F has an eigenvalue at $\lambda = 0$ of multiplicity (N-k).

The degeneracy of F = 1, for $\lambda = 0$, since the Jordan form of F has only one Jordan block associated with each eigenvalue. This fact follows from the cyclicity of F. This degeneracy property may be expressed:

$$\dim\{N[F-\lambda I_N^o] \mid \lambda = 0\} = \dim\{N[F]\} = 1$$

where $N[\cdot]$ = null space of a matrix.

The generalized eigenspace [13], $N[F-\lambda I_N^o]^{(N-k)}$, associated with $\lambda = 0$ has the following property:

$$\dim\{N[F-\lambda I_N^o]^{N-k} \mid \lambda = 0\} = \dim\{N[F]^{N-k}\} = N - k,$$

and further, for all $\alpha \geq N - k$:

$$\dim\{N[F-\lambda I_N^o]^{\alpha} \mid \lambda = 0\} = \dim\{N[F]^{\alpha}\} = N - k$$

Thus, $\rho\{[F-\lambda I]^{\alpha} \mid \lambda = 0\} = \rho\{F^{\alpha}\} = k$ for all $\alpha \geq N - k$. This is true since

$$\rho\{F^{\alpha}\} + \dim\{\mathcal{N}[F]^{\alpha}\} = \dim\{\mathcal{D}[F]^{\alpha}\} = N,$$

where $\mathcal{D}[\cdot]$ denotes the domain of a matrix.

$$\rho\{F^{\alpha}\} = k \text{ for all } \alpha \geq N - k \text{ implies}$$

$$\dim\{\mathcal{N}[F-\lambda I_N^o]^{\alpha} \mid \lambda = 0\} = N - k \quad \text{for all } \alpha \geq N - k.$$

This in turn implies $\lambda = 0$ is an eigenvalue of F with multiplicity $(N-k)$, and

$$\Delta(\lambda) = \lambda^{N-k}[\lambda^k - \sum_{i=1}^{N} a_i \lambda^{k-i}].$$

Thus, the characteristic equation requires $(a_N, \ldots, a_{k+1}) = 0$, $a_k \neq 0$. ■

By enumeration of F^k, $k = (1,2,\ldots,N)$, it is possible to demonstrate that:

$$F^k = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ & a^T & \\ & a^T F & \\ & a^T F^{k-1} & \end{bmatrix} + I_N^k \qquad \text{(A.2-7)}$$

When $k = N$ we have:

$$F^N = \begin{bmatrix} a^T \\ a^T\ F \\ \cdot \\ \cdot \\ \cdot \\ a^T\ F^{N-1} \end{bmatrix}, \tag{A.2-8}$$

since $I_N^N = 0$. Recognizing that F^N is the observability matrix for the pair (a^T, F), we have the following proposition:

Proposition A.2-2

The system (A.2-2, A.2-3) is completely observable if and only if $a_N \neq 0$.

Proof: It is sufficient to show that F^N is nonsingular if and only if $a_N \neq 0$.

$\Longrightarrow$ F^N nonsingular implies F^N has no eigenvalues $\lambda = 0$. In the context of Proposition A.2-1, k = N, and we have: F has no eigenvalues $\lambda = 0$, and $\rho\{F^\alpha\} = N$ for all $\alpha \geq 0$ imply $a_N \neq 0$.

$\Longleftarrow$ $a_N \neq 0$ implies the case in Proposition A.2-1 where k = N. Thus $a_N \neq 0$ implies F has no eigenvalues at $\lambda = 0$ and $\rho\{F^\alpha\} = N$ for all $\alpha \geq 0$. In particular, $\rho\{F^N\} = N$ and F^N is nonsingular. ∎

A.2.2 A Matrix Sequence Induced by the Input-Output Delay System

The 2N x 2N matrix, $\tilde{F}$, defining the dynamics of the I-O Delay state realization (2.2-8), induces the matrix sequence $\tilde{F}^k$, k = (0,1,2,. . .), as a result of the solution to the I-O Delay state difference equation. Imbedded in $\tilde{F}^k$ appears the following N x N matrix sequence, which we denote $\hat{F}(k)$:

$$F(k) = \sum_{j=0}^{k-1} F^{k-1-j} \begin{bmatrix} 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ & b^T & \end{bmatrix} I_N^j \tag{A.2-9}$$

where $b^T = [b_N \; . \; . \; . \; b_1]$ (A.2-10)

Note that $\hat{F}(o)$ is undefined, but we will adopt the notation $\hat{F}(o) = 0$ for completeness.

By enumeration of $\hat{F}(k)$, $k = (0,1,2,. \; . \; .)$, it is possible to show that:

$$\hat{F}(k) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b^T I_N^o + a^T \hat{F}(0) \\ b^T I_N^1 + a^T \hat{F}(1) \\ \vdots \\ b^T I_N^{k-1} + a^T \hat{F}(k-1) \end{bmatrix} \tag{A.2-11}$$

Alternatively, we may express the sequence $\hat{F}(k)$ recursively:

$$\hat{F}(k+1) = F \cdot \hat{F}(k) + \hat{F}(1) I_N^k \tag{A.2-12a}$$

$$= \hat{F}(k) I_N^1 + F^k \hat{F}(1) \tag{A.2-12b}$$

Of particular importance to the developments of Chapter 2 is the sequence element $\hat{F}(N)$:

$$\hat{F}(N) = \begin{bmatrix} b^T I_N^o & + & a^T\hat{F}(0) \\ b^T I_N^1 & + & a^T\hat{F}(1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b^T I_N^{N-1} & + & a^T\hat{F}(N-1) \end{bmatrix} \tag{A.2-13}$$

Using the $\hat{F}(k)$ notation, we may express the sequence $\tilde{F}^k$ as

$$\tilde{F}^k = \left[\begin{array}{c|c} F^k & \hat{F}(k) \\ \hline 0 & I_N^k \end{array}\right] \tag{A.2-14}$$

In particular, for k = N,

$$\tilde{F}^N \quad \left[\begin{array}{c|c} F^N & \hat{F}(N) \\ \hline 0 & 0 \end{array}\right] \tag{A.2-15}$$

The observability matrix for the I-O Delay state realization may also be expressed in a very compact form:

$$R_o = \begin{bmatrix} p^T \\ p^T\tilde{F} \\ \cdot \\ \cdot \\ p^T\tilde{F}^{N-1} \\ (p^T)\tilde{F}^N \\ (p^T\tilde{F})\tilde{F}^N \\ \cdot \\ \cdot \\ \cdot \\ (p^T\tilde{F}^{N-1})\tilde{F}^N \end{bmatrix} = \left[\begin{array}{c|c} \begin{matrix} a^T \\ a^T F \\ \cdot \\ \cdot \\ a^T F^{N-1} \end{matrix} & \begin{matrix} b^T I_N^o + a^T\hat{F}(0) \\ b^T I_N^1 + a^T\hat{F}(1) \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ b^T I_N^{N-1} + a^T\hat{F}(N-1) \end{matrix} \\ \hline \multicolumn{2}{c}{\left[\begin{array}{c|c} a^T & b^T I_N^o + a^T\hat{F}(0) \\ a^T F & b^T I_N^1 + a^T\hat{F}(1) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a^T F^{N-1} & b^T I_N^{N-1} + a^T\hat{F}(N-1) \end{array}\right] \cdot \tilde{F}^N} \end{array}\right] =$$

$$= \left[\begin{array}{c|c} F^N & \hat{F}(N) \\ \hline [F^N \mid \hat{F}(N)] \cdot \tilde{F}^N & \end{array}\right]$$

Thus, using (A.2-15)

$$R_o = \begin{bmatrix} F^N & \hat{F}(N) \\ & \\ F^{2N} & F^N\hat{F}(N) \end{bmatrix} \qquad \text{(A.2-16)}$$

VITA

Dale Keith Barbour was born in Buffalo, New York, on August 20, 1945. He received the Sc.B degree in Engineering from Brown University in June 1967, and the MSEE degree from Carnegie-Mellon University in June 1969. From June 1967, to September 1972, he was employed by the U.S. Naval Underwater Systems Center, Newport, Rhode Island. From September 1972, until the present, he has been a Research Assistant at the Coordinated Science Laboratory at the University of Illinois.